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CENTRAL LIMIT THEOREM FOR STATIONARY PRODUCTS OF TORAL AUTOMORPHISMS

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Abstract. Let $(A_n(\omega))$ be a stationary process in $\mathcal{M}_d^*(\mathbb{Z})$. For a Hölder function f on $\mathbb{T}^d$ we consider the sums $\sum_{k=1}^n f({}^t\!A_k(\omega)\,{}^t\!A_{k-1}(\omega)\cdots{}^t\!A_1(\omega)\,x \bmod 1)$ and prove a Central Limit Theorem for a.e. ω in different situations in particular for "kicked" stationary processes. We use the method of multiplicative systems of Komlòs and the Multiplicative Ergodic Theorem.

1. **Introduction.** Let $(M_n)_{n\geq 1}$ be a sequence in the set $\mathcal{M}_d^*(\mathbb{Z})$ of $d\times d$ non singular matrices with coefficients in $\mathbb{Z}$. It defines a sequence of endomorphisms of the torus. The general question of the central limit theorem (CLT) for $S_n f = \sum_{k=1}^n f({}^t\!M_k .)$, for a regular real function f on $\mathbb{T}^d$, covers different particular cases. If $d = 1$, it corresponds to arithmetic sums $\sum_{k=0}^{n-1} f(q_k x)$ and, for lacunary sequences of positive integers (q_n), it has been studied by several authors (Fortet, Kac, Salem, Zygmund, Gaposhkin [11], Berkes [5], recently Berkes and Aistleitner [1]).

Another situation is for $d > 1$ the action on $\mathbb{T}^d$ of a product ${}^t\!M_k = {}^t A_1 ... {}^t\!A_k$, with $A_k \in \mathcal{M}_d^*(\mathbb{Z})$. The sequence of maps obtained by composition of the transformations[1] $\tau_n x = {}^t\!A_n x \bmod 1$ can be viewed as a non autonomous or "sequential" dynamical system.

Analogous examples of sequential dynamical systems on a probability space have been studied, for example in [17] for transformations chosen at random in the neighborhood of a given one, in [4] for a non perturbative case with geometrical assumptions on the transformations, in [9] for expanding maps of the interval.

Here we will mainly consider different examples of stationary, not necessarily independent, processes $(A_k(\omega))$ in $\mathcal{M}_d^*(\mathbb{Z})$ and address the question of the CLT with respect to the Lebesgue measure λ on $\mathbb{T}^d$ for almost every ω (and the non degeneracy of the limit law) for

$$S_n f(\omega, x) = \sum_{k=1}^n f({}^t\!A_k(\omega)\,{}^t\!A_{k-1}(\omega)\cdots\,{}^t\!A_1(\omega)x \bmod 1). \tag{1}$$

In Section 2 we give sufficient conditions on (A_n) for the convergence of the distribution of $\frac{1}{\|S_n\|_2} S_n f$ toward a normal law with a small rate. The proof is based on

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[1]For simplicity of notations, it is convenient to act on the torus by transposed matrices.

the method of "multiplicative systems" (cf. Komlòs [12]). We give also an example where a coboundary condition leads to a non standard normalisation.

Section 3 is devoted to the stationary case and the question of the CLT for a.e. ω for the sums (1). We consider an ergodic dynamical system (Ω, μ, θ), where θ is an invertible measure preserving transformation on a probability space (Ω, μ) and the skew product defined on $(\Omega \times \mathbb{T}^d, \mu \times \lambda)$ by $\theta_\tau : (\omega, x) \mapsto (\theta\omega, {}^tA(\omega)x \bmod 1)$, where $\omega \to A(\omega)$ is a measurable map from Ω to a finite set $\mathcal{A}$ of matrices in $\mathcal{M}_d^*(\mathbb{Z})$. Under some conditions we obtain a strong mixing property for this skew product and show that for regular functions the variance exists and is not zero.

The abstract results are then applied to explicit models. When the elements of $\mathcal{A}$ are 2×2 positive matrices, the CLT holds for every sequence under a variance condition. In particular the CLT holds for a.e. ω in the stationary case. There the invariant positive cone plays an essential role like in [2] where analogous problems have been studied. In dimension 2 we consider also "kicked systems" introduced by Polterovich and Rudnick, who proved a stable mixing property for this model. For stationary kicked processes, which can be viewed as a perturbation of the iteration of a single automorphism, we obtain a CLT for a.e. ω. In a last subsection we show the non nullity of the variance for "stationary" arithmetic sums in dimension 1.

2. **Multiplicative systems and CLT.**

2.1. **Preliminaries, a criterion of Komlòs for multiplicative systems.**

Notation. Let d be an integer ≥ 2 and $\|\cdot\|$ be the norm on $\mathbb{R}^d$ defined by $\|x\| = \max_{1\leq i\leq d} |x_i|, x \in \mathbb{R}^d$. We denote by $\delta(x, y) := \inf_{n\in\mathbb{Z}^d} \|x - y - n\|$ the distance on the torus. The characters on $\mathbb{T}^d$ are $\chi_n : x \to \chi(n, x) := e^{2\pi i\langle n,x\rangle}$, $n = (n_1, ..., n_d) \in \mathbb{Z}^d$. Often C will denote a "generic" constant which may change from a line to the other.

The Lebesgue measure λ on $\mathbb{T}^d$ is invariant by surjective endomorphisms of the torus. The space of functions f in $L^2(\mathbb{T}^d, \lambda)$ such that $\lambda(f) = 0$ is denoted by $L^2_0(\mathbb{T}^d)$ and the Fourier coefficients of $f \in L^2(\mathbb{T}^d)$ by $\hat{f}(p)$ or f_p, $p \in \mathbb{Z}^d$. The degree of a trigonometric polynomial g on $\mathbb{T}^d$ is less than D (notation $\deg(g) \leq D$), if $\hat{g}(p) = 0$, for $\|p\| > D$. In what follows all trigonometric polynomials will be centered.

We denote by $\mathcal{H}_0(\mathbb{T}^d)$ the space of bounded functions f in $L^\infty(\mathbb{T}^d)$ with null integral such that, for a constant C and $\alpha \in]0, 1]$, $\|f(. - t) - f(.)\|_1 \leq C\|t\|^\alpha$, $\forall t \in \mathbb{T}^d$.

The α-Hölder functions for some $\alpha \in]0, 1]$, as well as the characteristic function of regular sets belong to $\mathcal{H}_0(\mathbb{T}^d)$ (a subset $E \subset \mathbb{T}^d$ is *regular* if there exists $C > 0$ and $\alpha \in]0, 1]$ such that $\lambda(\{x \in \mathbb{T}^d : \delta(x, \partial E) \leq \varepsilon\}) \leq C\varepsilon^\alpha, \forall\varepsilon > 0$.) Therefore the statements about functions in $\mathcal{H}_0(\mathbb{T}^d)$ below are valid in particular for the usual Hölder functions.

We will use the following approximation result:

Proposition 1. *For every $f \in \mathcal{H}_0(\mathbb{T}^d)$, there exist $\alpha \in]0, 1]$ and a sequence of trigonometric polynomials g_n such that* $\deg(g_n) \leq n$, $\|g_n\|_\infty \leq \|f\|_\infty$, $\|g_n\|_2 \leq \|f\|_2$ *and* $\|g_n - f\|_2 = O(n^{-\alpha})$.

Proof. a) Let $K_n(t) = K_n(t_1)...K_n(t_d)$ be the Fejér kernel in dimension $d \geq 1$, where $K_n(t_1) = \frac{1}{n} \frac{\sin^2(\pi n t_1)}{\sin^2(\pi t_1)}$. For every $\beta \in]0, 1]$, there exists a constant C such that

$$\begin{aligned} \int K_n(t) \, \|t\|^\beta \, dt &= [\int_{\|t\| \geq \frac{1}{n}} + \int_{\|t\| < \frac{1}{n}}] \, K_n(t) \, \|t\|^\beta \, dt \\ &\leq \frac{C}{n} \sum_{i=1}^{d} \int_{|t_i| \geq \frac{1}{n}} |t_i|^{\beta-2} \, dt_i + n^{-\beta} \\ &\leq \frac{2Cd}{n} \frac{n^{1-\beta}}{1-\beta} + n^{-\beta} = O(n^{-\beta}). \end{aligned}$$

For f in $\mathcal{H}_0$, $K_n * f$ is a trigonometric polynomial of degree $\leq n$ such that $\|K_n * f\|_\infty \leq \|f\|_\infty$, $\|K_n * f\|_2 \leq \|f\|_2$. There is $\beta \in]0, 1]$ such that:

$$\begin{aligned} \|K_n * f - f\|_1 &\leq \int \int K_n(t) \, |f(x-t) - f(x)| \, dt \, \lambda(dx) \\ &= \int K_n(t) \, \|f(. - t) - f(.)\|_1 \, dt \\ &= O(\int K_n(t) \, |t|^\beta \, dt) = O(n^{-\beta}). \end{aligned}$$

Therefore we obtain, with $\alpha = \beta/2$: $\|K_n * f - f\|_2 \leq (2\|f\|_\infty)^{\frac{1}{2}} \, \|K_n * f - f\|_1^{\frac{1}{2}} = O(n^{-\alpha})$. $\square$

Recall the notation $\mathcal{M}_d^*(\mathbb{Z})$ for the set of $d \times d$ invertible matrices with coefficients in $\mathbb{Z}$. In what follows, (M_k) is a sequence in $\mathcal{M}_d^*(\mathbb{Z})$. For a function f on $\mathbb{T}^d$ we denote by $S_n f(x)$ or simply $S_n(x)$ the sums $S_n f(x) = \sum_{k=1}^n f({}^t\!M_k x)$.

A special case (product case) is when M_k is a product: $M_k = A_1 ... A_k$, where (A_k) is a sequence in $\mathcal{M}_d^*(\mathbb{Z})$. To the action of a product of matrices in $\mathcal{M}_d^*(\mathbb{Z})$ on $\mathbb{T}^d$ corresponds a dual action on the characters by the transposed matrices with *composition on the right side*. For simplicity of notations, we choose to act on $\mathbb{T}^d$ by the transposed matrices ${}^t\!M_k$. For $j \geq i \geq 0$, with the convention $A_0 = A_0^0 = \mathrm{Id}$, $A_0^j = A_1^j$, we write

$$A_i^j := A_i \ldots A_j. \tag{2}$$

A function f in $L_0^2(\mathbb{T}^d)$ satisfies the *decorrelation property*, if there are constants $C(f)$ and $0 < \kappa(f) < 1$ such that

$$| \int_{\mathbb{T}^d} f({}^t\!M_{\ell+r} x) \, \overline{f({}^t\!M_\ell x)} \, dx | \leq C(f) \, \kappa(f)^r, \, \forall r, \ell \geq 0. \tag{3}$$

A criterion of Komlòs

In the proof of the central limit theorem for products of toral automorphisms we use the following lemma on "multiplicative systems" (cf. Komlòs [12]) (see [14] for another application of this method). The quantitative formulation of the result yields a rate of convergence in the CLT.

Lemma 2.1. *Let u be an integer ≥ 1 and a be a real positive number. Let $(\zeta_k)_{0 \leq k \leq u-1}$ be a sequence of length u of real bounded random variables defined* *on a probability space* (X, λ). *Let us denote, for* $t \in \mathbb{R}$*:*

$$\begin{aligned} Z(t,.) &= \exp(it \sum_{k=0}^{u-1} \zeta_k(.)),\ Q(t,.) = \prod_{k=0}^{u-1}(1+it\zeta_k(.)), \\ Y &= \sum_{k=0}^{u-1} \zeta_k^2,\quad \delta = \max_{0\le k\le u-1} \|\zeta_k\|_\infty. \end{aligned}$$

Under the conditions $|t|\,\delta \le 1$, $|t|\|Y-a\|_2^{\frac12} \le 1$, $\lambda[Q(t,.)] \equiv 1$, $\|Q(t,.)\|_2 = O(e^{\frac12 a\, t^2})$, *there is a constant* C *such that*

$$|\lambda[Z(t,.)] - e^{-\frac12 a\, t^2}| \le C(u\,|t|^3\delta^3 + |t|\|Y-a\|_2^{\frac12}). \tag{4}$$

Proof. 1) Setting $\psi(y) = (1+iy)e^{-\frac12 y^2}e^{-iy} = \rho(y)e^{i\theta(y)}$, where $\rho(y) = |\psi(y)|$, we have

$$\ln\rho(y) = \frac12[\ln(1+y^2) - y^2] \le 0,\ \tan(\theta(y)) = \frac{y-\tan y}{1+y\tan y}.$$

An elementary computation gives the upper bounds

$$|\ln\rho(y)| \le \frac14|y|^4,\ |\theta(y)| = O(|y|^3),\ \forall y \in [-1,1]. \tag{5}$$

Let us write: $Z(t,.) = Q(t,.)\ \exp(-\frac12 t^2\ Y)\ [\prod_{k=0}^{u-1}\psi(t\zeta_k)]^{-1}$. As $\ln\rho(t\zeta_k) \le 0$, we have:

$$\begin{aligned} &|Z(t,.) - Q(t,.)\ \exp(-\frac12 t^2\ Y)| = |Z(t,.) - Z(t,.)\prod_{k=0}^{u-1}\psi(t\zeta_k)| \\ &= |1 - \prod_{k=0}^{u-1}\psi(t\zeta_k)| \le |1 - e^{\sum_{k=0}^{u-1}\ln\rho(t\zeta_k)}| + |1 - e^{i\sum_{k=0}^{u-1}\theta(t\zeta_k)}| \\ &\le \sum_{k=0}^{u-1}|\ln\rho(t\zeta_k)| + \sum_{k=0}^{u-1}|\theta(t\zeta_k)|. \end{aligned}$$

If $|t|\delta \le 1$, where $\delta = \max_k \|\zeta_k\|_\infty$, we can apply (5) and obtain for a constant C:

$$|Z(t,.) - Q(t,.)\ e^{-\frac12 t^2\, Y}| \le C|t|^3 \sum_{k=0}^{u-1}|\zeta_k|^3 \le Cu|t|^3\delta^3. \tag{6}$$

2) For $0 \le \varepsilon \le 1$, let $A_\varepsilon(t) = \{x : t^2|Y(x) - a| \le \varepsilon\}$. Using (6) and $\lambda[Q(t,.)] \equiv 1$, we get

$$\begin{aligned} |\lambda[Z(t,.) - e^{-\frac12 a\, t^2}]| &\le |\lambda[1_{A_\varepsilon(t)}(Z(t,.) - Q(t,.)e^{-\frac12 a\, t^2})]| + 2\lambda(A_\varepsilon^c(t)) \\ &\le C\,u\,|t|^3\delta^3 + \lambda[1_{A_\varepsilon(t)}\,e^{-\frac12 a\, t^2}|Q(t,.)\,[e^{-\frac12 t^2\,(Y-a)} - 1]|)] \\ &\qquad +2\lambda(A_\varepsilon^c(t)). \end{aligned}$$

From the inequality $|1-e^s| \le (e-1)|s| \le 2|s|, \forall s \in [-1,1]$, we have

$$\|1_{A_\varepsilon(t)}\,[e^{-\frac{t^2}{2}(Y-a)} - 1]\|_2 \le \varepsilon.$$

Choosing $\varepsilon = |t|\|Y-a\|_2^{\frac12}$, we get $\lambda(A_\varepsilon^c(t)) \le \varepsilon^{-2}t^4\|Y-a\|_2^2 \le t^2\|Y-a\|_2$; hence

$$\begin{aligned} |\lambda[1_{A_\varepsilon(t)}(Z(t,.) - e^{-\frac12 a\, t^2}]| &\le C\,[u\,|t|^3\delta^3 + |t|\,e^{-\frac12 a\, t^2}\,\|Q(t,.)\|_2\,\|Y-a\|_2^{\frac12} \\ &\qquad + t^2\|Y-a\|_2]. \end{aligned}$$

Thus, with the assumptions of the lemma, we obtain (4). □

2.2. **Separation of frequencies and growth of the matrices.** In order to apply Lemma 2.1 to $S_n f(x) = \sum_{k=1}^n f({}^t\!M_k x)$, we need a property of "separation of frequencies" which is expressed in the following property.

Property 1. *Let D, Δ be positive reals. We say that the property $\mathcal{S}(n, D, \Delta)$ holds for a set $(M_1, \dots, M_n)$ of $n \geq 1$ matrices in $\mathcal{M}_d^*(\mathbb{Z})$ if the following condition is satisfied:*

Let s be an integer ≥ 1. Let $1 \leq \ell_1 \leq \ell_1' < \ell_2 \leq \ell_2' < ... < \ell_s \leq \ell_s' \leq n$ be any increasing sequence of $2s$ integers, such that $\ell_{j+1} \geq \ \ell_j' + \Delta$ for $j = 1, ..., s-1$. Then for every vectors $p_1, ..., p_s$ and $p_1', ..., p_s'$ in $\mathbb{Z}^d$ such that $\|p_j\|, \|p_j'\| \leq D$, for $j = 1, ..., s$, we have

$$M_{\ell_s'} p_s' + M_{\ell_s} p_s \neq 0 \Rightarrow \sum_{j=1}^s [M_{\ell_j'} p_j' + M_{\ell_j} p_j] \neq 0. \tag{7}$$

Property $\mathcal{S}(n, D, \Delta)$ for $(M_1, \dots, M_n)$ implies in particular the following. Let $\ell_1 < \ell_2 < ... < \ell_s \leq n$ be an increasing sequence of s integers such that $\ell_{j+1} \geq \ \ell_j + \Delta$ for $j = 1, ..., s-1$; then, for every family $p_1, ..., p_s \in \mathbb{Z}^d$ such that $\|p_j\| \leq D$ for $j = 1, ..., s$,

$$p_s \neq 0 \Rightarrow \sum_{j=1}^s M_{\ell_j} p_j \neq 0. \tag{8}$$

Conditions on the growth of $\|M_n p\|$

We introduce conditions on the growth of M_n. Condition 1 ensures a decorrelation property. Condition 2 (or Inequality (11) for products) is used for the separation of frequencies property. Condition 3, which is uniform with respect to the choice of the blocks, is satisfied by two families of examples, matrices in $\mathrm{SL}(2, \mathbb{Z}^+)$ and "kicked" processes.

Condition 1. *There is $C_1 > 0$ such that, for every $D \geq 1$, for every $q, p \in \mathbb{Z}^d \setminus \{0\}$ with $\|q\|, \|p\| \leq D$,*

$$M_{\ell+r}\, q \neq M_\ell\, p, \forall r > C_1 \ln D, \forall \ell \geq 0. \tag{9}$$

In the product case, i.e. when $M_n = A_1 ... A_n$, (9) reads:

$$A_\ell^{\ell+r} q \neq p, \forall r > C_1 \ln D, \forall \ell \geq 0. \tag{10}$$

Condition 2. *($M_n = A_1 ... A_n$) There are constants $\gamma > 1$ and $C_1, c > 0$ such that*

$$\|A_1^\ell q\| \ \geq c \gamma^r \|A_1^{\ell - r}\|, \forall\, \ell \geq r \geq C_1 \ln \|q\|, \ \forall q \in \mathbb{Z}^d \setminus \{0\}. \tag{11}$$

Condition 3. *There are constants $\gamma > 1$ and $\delta, C_1, c > 0$ such that*

$$\forall A_1, ..., A_r \in \mathcal{A}, \|A_1 ... A_r q\| \geq c\, \|q\|^{-\delta} \gamma^r, \forall r > C_1 \ \ln \|q\|, \ \forall q \in \mathbb{Z}^d \setminus \{0\}. \tag{12}$$

Note that (11) and (12) imply (10). The following condition is a reinforcement of the previous conditions and expresses the "superlacunarity" of the sequence (M_n).

Condition 4. *There are positive constants δ, C_1, c and a sequence $(\gamma_\ell)_{\ell \geq 1}$ of numbers > 1 with $\lim_\ell \gamma_\ell = +\infty$ such that, for every $q \in \mathbb{Z}^d \setminus \{0\}$,*

$$\|M_{\ell+r} q\| \ \geq c \gamma_\ell^r \|M_\ell\|, \forall\, \ell \geq 1, \forall\, r \geq C_1 \ln \|q\|. \tag{13}$$

In the scalar case $d = 1$, the matrices M_n are numbers q_n. This corresponds to the hypothesis $\lim_n q_{n+1}/q_n = +\infty$. Condition 4 implies the following one:

Condition 5. *There is a sequence $(c_2(\ell))_{\ell \geq 1}$ of positive numbers with $\lim_\ell c_2(\ell) = 0$ such that, for every $D \geq 1$, for every $q, p \in \mathbb{Z}^d \setminus \{0\}$ with $\|q\|, \|p\| \leq D$,*

$$M_{\ell+r}\, q \neq M_\ell\, p,\ \forall \ell \geq 1,\ \forall r > c_2(\ell) \ln D. \tag{14}$$

Decorrelation

The next proposition shows that Condition 1 implies the decorrelation property (3).

Proposition 2. *1) Assume Condition 1 (i.e. (9) or, in the product case, (10)). If g is a trigonometric polynomial of degree D, then we have:*

$$\int_{\mathbb{T}^d} g({}^t\!M_{\ell+r} x)\, \overline{g({}^t\!M_\ell x)}\, dx = 0, \forall r \geq C_1 \ln D, \forall \ell \geq 0.$$

For $f, f' \in \mathcal{H}_0(\mathbb{T}^d)$ there are constants C and $\kappa < 1$ such that

$$|\int_{\mathbb{T}^d} f({}^t\!M_\ell x)\, f'({}^t\!M_{\ell+r} x)\, dx| \quad \leq \quad C\kappa^r,\ \forall r, \ell \geq 0, \tag{15}$$

$$\|S_n f\|_2^2 \quad = \quad \|\sum_{k=1}^{n} f({}^t\!M_k .)\|_2^2 = O(n). \tag{16}$$

2) If Condition 5 holds, then for every $f \in \mathcal{H}_0(\mathbb{T}^d)$

$$\lim_n \frac{1}{n} \|S_n f\|_2^2 = \|f\|_2^2. \tag{17}$$

Proof. 1) Let $g(x) = \sum_{0 < \|p\| \leq D} g_p \chi(p, x)$ be a trigonometric polynomial of degree $D \geq 1$. By Condition 1 we have for all $\ell \geq 0$:

$$\begin{aligned} \langle g \circ {}^t\!M_\ell, g \circ {}^t\!M_{\ell+r} \rangle_\lambda &= \sum_{0 < \|p\|, \|q\| \leq D} \int\!\!\int g_p \chi(M_{\ell+r} p, x)) (\overline{g_q \chi(M_\ell q, x)})\ dx \\ &= \sum_{0 < \|p\|, \|q\| \leq D} g_p\, \overline{g_q}\ 1_{M_{\ell+r} p = M_\ell q}\ = 0, \forall r \geq C_1 \ln D. \end{aligned}$$

Let $f \in \mathcal{H}_0(\mathbb{T}^d)$. Let $c_1 > 1$ be such that $\ln c_1 < 1/C_1$. Proposition 1 shows that there exist $C = C(f)$, $\alpha \in]0,1]$ and for every $r \geq 1$ a trigonometric polynomial g_r with $\deg(g_r) \leq c_1^r$ such that $\|g_r - f\|_2 \leq C c_1^{-\alpha r}$. The choice of c_1 implies $\langle g_r \circ {}^t\!M_{\ell+r}, g_r \circ {}^t\!M_\ell \rangle = 0$. Hence

$$\begin{aligned} &|\langle f \circ {}^t\!M_{\ell+r}, f \circ {}^t\!M_\ell \rangle| \leq |\langle (f - g_r) \circ {}^t\!M_{\ell+r}, f \circ {}^t\!M_\ell \rangle| \\ &\quad + |\langle g_r \circ {}^t\!M_{\ell+r}, g_r \circ {}^t\!M_\ell \rangle| + |\langle g_r \circ {}^t\!M_{\ell+r}, (f - g_r) \circ {}^t\!M_\ell \rangle| \leq 2C \|f\|_2 c_1^{-\alpha r}. \end{aligned}$$

Therefore we get (15) with $\kappa = c_1^{-\alpha}$ when $f' = f$. In the same way we obtain (15) for f, f' in $\mathcal{H}_0(\mathbb{T}^d)$. For the variance we have

$$\begin{aligned} \frac{1}{n} \|S_n f\|_2^2 &= \frac{1}{n} \sum_{\ell=0}^{n-1} \sum_{\ell'=0}^{n-1} \int_{\mathbb{T}^d} f({}^t\!M_\ell\, x)\, f({}^t\!M_{\ell'}\, x)\, dx \\ &= \|f\|^2 + \frac{2}{n} \sum_{r=1}^{n-1} \sum_{\ell=0}^{n-1-r} \int_{\mathbb{T}^d} f({}^t\!M_\ell\, x)\, f({}^t\!M_{\ell+r}\, x)\, dx \\ &\leq \|f\|_2^2 + 2C(f) \|f\|_2 \sum_{r=1}^{n-1} (1 - \frac{r}{n}) \kappa^r \leq \|f\|_2^2 + \frac{2C(f)\kappa}{1 - \kappa} \|f\|_2 < +\infty. \end{aligned}$$

2) Let us consider the superlacunary case and suppose (14) of Condition 5. Let $f \in \mathcal{H}_0(\mathbb{T}^d)$. Let $\theta(\ell) > 1$ with $\lim_\ell \theta(\ell) = +\infty$ be such that $\ln \theta(\ell) < 1/c_2(\ell)$, where $c_2(\ell)$ is given by (14). Proposition 1 shows that there is $\alpha \in]0,1]$ such that, for every $r \geq 1$, there exists a trigonometric polynomial g_r of degree less than $\theta(\ell)^r$ such that $\|g_r - f\|_2 \leq C(f)\,\theta(\ell)^{-\alpha r}$. The choice of $\theta(\ell)$ implies $\langle g_r \circ {}^tM_{\ell+r}, g_r \circ {}^tM_\ell\rangle = 0$. Hence

$$\begin{aligned} &|\langle f \circ {}^tM_{\ell+r}, f \circ {}^tM_\ell\rangle| \leq |\langle (f-g_r) \circ {}^tM_{\ell+r}, f \circ {}^tM_\ell\rangle| \\ &\qquad + |\langle g_r \circ {}^tM_{\ell+r}, g_r \circ {}^tM_\ell\rangle| + |\langle g_r \circ {}^tM_{\ell+r}, (f-g_r) \circ {}^tM_\ell\rangle| \\ \leq\ & 2C(f)\,\theta(\ell)^{-\alpha r}. \end{aligned}$$

Thus we have, since $\lim_\ell \theta(\ell) = +\infty$,

$$\begin{aligned} |\frac{1}{n}\|S_n f\|_2^2 - \|f\|^2| &\leq \frac{2}{n}\sum_{r=1}^{n-1}\sum_{\ell=0}^{n-1-r} 2C(f)\,\theta(\ell)^{-\alpha r} \\ &\leq 4C(f)\,\frac{1}{n}\sum_{\ell=0}^{n-1}\frac{\theta(\ell)^{-\alpha}}{1-\theta(\ell)^{-\alpha}} \underset{n\to\infty}{\to} 0. \end{aligned}$$

□

For further use we give another consequence of Condition 1.

Proposition 3. *Assume Condition 1. For $J \subset I \subset \mathbb{N}$, denote by $S_n^I f, S_n^J f$ the sums*

$$S_n^I f = \sum_{k\in[1,n]\cap I} f({}^tA_1^k x),\ \ S_n^J f = \sum_{k\in[1,n]\cap J} f({}^tA_1^k x).$$

Then, for $f \in \mathcal{H}_0(\mathbb{T}^d)$, $\|S_n^I f - S_n^J f\|_2^2 \leq O(\mathrm{Card}([1,n]\cap J^c))$.

Proof. Using (15), we get

$$\begin{aligned} \|S_n^I - S_n^J\|_2^2 &= \sum_{k,k'\in I\cap J^c\cap[1,n]} |\int f({}^tA_1^k x) f({}^tA_1^{k'} x)\, dx| \\ &\leq C(f)\,\|f\|_2 \sum_{k\in I\cap J^c\cap[1,n]} [\sum_{k'\in I\cap J^c\cap[1,n]} \kappa^{|k-k'|}] \\ &= O(\mathrm{Card}([1,n]\cap J^c)). \end{aligned}$$

□

Separation of frequencies

Proposition 4. *(Product case) Under Condition 2 for $(A_1^\ell)_{\ell=1,\dots,n}$, there is a constant C_S such that $\mathcal{S}(n,D,\Delta)$ holds if $\Delta \geq C_S \ln D$.*

Proof. Let $\rho > 0$ be such that $c^{-1}D\,\gamma^{-\rho} < 1/2$ and $\rho \geq C_1 \ln D$, i.e

$$\rho > \max(\frac{\ln(2c^{-1}D)}{\ln\gamma}, C_1 \ln D).$$

Let $C_2 := \ln \max_{A\in\mathcal{A}} \|A\|$. Recall that the constants γ, c and C_1 were introduced in Condition (2). In the proof we will need that Δ satisfies the inequalities

$$\Delta \geq C_1(C_2\rho + \ln D) \text{ and } \frac{2c^{-1}D}{(1-\gamma^{-\Delta})}\gamma^{-\Delta} < \frac{1}{2}. \tag{18}$$

This is equivalent to $\Delta > \max(\frac{\ln(1+4c^{-1}D)}{\ln\gamma}, C_1(C_2\rho + \ln D))$. There is a constant C_S depending only on c, C_1, C_2 such that Δ satisfies (18) if $\Delta \geq C_S \ln D$.

Now we show that the separation property holds if Δ satisfies (18). We use the notations of Condition $\mathcal{S}(n, D, \Delta)$. For $s \geq 1$, let $1 \leq \ell_1 \leq \ell_1' < \ell_2 \leq \ell_2' < ... < \ell_s \leq \ell_s' \leq n$ be a sequence of $2s$ integers, such that $\ell_{j+1} \geq \ \ell_j' + \Delta$ for $j = 1, ..., s-1$. Let $p_1, p_2, ..., p_s,\ p_1', p_2', ..., p_s'$ be vectors such that $\|p_j\|, \|p_j'\| \leq D$, $j = 1, ..., s$. We have to show that the equation

$$A_1^{\ell_s'} p_s' + A_1^{\ell_s} p_s + \sum_{j=1}^{s-1} [A_1^{\ell_j'} p_j' + A_1^{\ell_j} p_j] = 0, \tag{19}$$

with $A_1^{\ell_s'} p_s' + A_1^{\ell_s} p_s \neq 0$ is never satisfied. Assume the contrary. We can suppose $p_s' \neq 0$ (otherwise $p_s \neq 0$ and we consider $\|A_1^{\ell_s} p_s\|$ instead of $\|A_1^{\ell_s'} p_s'\|$).

Write $q_s = A_{\ell_s+1}^{\ell_s'} p_s' + p_s$. We have $1 \leq \|q_s\| \leq 2De^{C_2(\ell_s' - \ell_s)}$.

1) Assume $\ell_s' - \ell_s \leq \rho$. Then, by (18), we have $\Delta \geq C_1[C_2(\ell_s' - \ell_s) + \ln D]$, so that we can apply (11) (replacing q by q_s in (11)), and obtain for $\ell = \ell_j, \ell_j'$ and $p = p_j, p_j', j = 1, ..., s-1$, since $\ell_s - \ell_j, \ell_s - \ell_j' \geq \Delta$:

$$\|A_1^{\ell} p\| \leq D\|A_1^{\ell}\| \leq Dc^{-1}\gamma^{-(\ell_s - \ell)}\|A_1^{\ell_s} q_s\|, \text{ for } \ell = \ell_j, \ell_j', p = p_j, p_j', 1 \leq j \leq s-1;$$

therefore

$$\begin{aligned} &\|\sum_{j=1}^{s-1}[A_1^{\ell_j'} p_j' + A_1^{\ell_j} p_j]\| \leq c^{-1} D \|A_1^{\ell_s} q_s\| \ \sum_{j=1}^{s-1} [\gamma^{-(\ell_s - \ell_j')} + \gamma^{-(\ell_s - \ell_j)}] \\ &\leq 2c^{-1} D \ [\sum_{j=1}^{s-1} \gamma^{-j\Delta}] \ \|A_1^{\ell_s} q_s\| \leq \frac{2c^{-1}D}{(1-\gamma^{-\Delta})} \gamma^{-\Delta} \ \|A_1^{\ell_s} q_s\| < \frac{1}{2}\|A_1^{\ell_s} q_s\|. \end{aligned}$$

We obtain a contradiction, since by Equation (19):

$$\|A_1^{\ell_s} q_s\| \quad = \quad \|A_1^{\ell_s'} p_s' + A_1^{\ell_s} p_s\| = \|\sum_{j=1}^{s-1}[A_1^{\ell_j'} p_j' + A_1^{\ell_j} p_j]\|.$$

2) Now consider the case $\ell_s' - \ell_s \geq \rho$. Then, since $\ell_s' - \ell_s \geq \rho \geq C_1 \ln D$, we have: $\|A_1^{\ell_s} p_s\| \leq c^{-1} D\, \gamma^{-(\ell_s' - \ell_s)} \ \|A_1^{\ell_s'} p_s'\|$ and

$$\begin{aligned} \|A_1^{\ell_s} p_s \quad &+ \quad \sum_{j=1}^{s-1} A_1^{\ell_j'} p_j' + \sum_{j=1}^{s-1} A_1^{\ell_j} p_j\| \\ &\leq \quad c^{-1} D\, [\gamma^{-(\ell_s' - \ell_s)} + \sum_{j=1}^{s-1} \gamma^{-(\ell_s' - \ell_j')} + \sum_{j=1}^{s-1} \gamma^{-(\ell_s' - \ell_j)}] \, \|A_1^{\ell_s'} p_s'\| \\ &\leq \quad [c^{-1} D \gamma^{-(\ell_s' - \ell_s)} + \frac{2c^{-1}D}{(1-\gamma^{-\Delta})}\gamma^{-\Delta}] \ \|A_1^{\ell_s'} p_s'\| < \|A_1^{\ell_s'} p_s'\|. \end{aligned}$$

Hence again a contradiction, since by (19): $\|A_1^{\ell_s'} p_s'\| = \|A_1^{\ell_s} p_s + \sum_{j=1}^{s-1}[A_1^{\ell_j'} p_j' + A_1^{\ell_j} p_j]\|$. $\square$

Remark 1. The previous result is valid for the "product case". In the general case of a sequence (M_n) the lacunarity condition analogous to Condition 2 does not imply the frequency separation property, even for the one dimensional case when $(M_n) = (q_n)$ is an increasing sequence of positive integers such that $\inf_{n>1} q_{n+1}/q_n > 1$ (for the counter example of Fortet and Kac $q_n = 2^n - 1$, see [1], [7]). The superlacunarity growth condition of q_n is a sufficient condition and this extends easily in dimension $d > 1$.

Proposition 5. *(Non product case) Under Condition 4 for $(M_1, \ldots, M_n)$ there is a constant C_S such that $\mathcal{S}(n, D, \Delta)$ holds if $\Delta \geq C_S \ln D$.*

2.3. **Application to the CLT.** Now we focus on the characteristic function $t \to \lambda[e^{it\frac{S_n}{\|S_n\|_2}}]$ for a real trigonometric polynomial g. Recall that $S_n(x) = S_n g(x) = \sum_{k=1}^n g({}^t\!M_k x)$. We suppose that $\|S_n\|_2 \neq 0$. First we give the general bound (20). When the sums $\|S_n\|_2$ are of order $\sqrt{n}$, it implies (21) or (22) from which a rate of convergence toward the normal law can be deduced (see also 2.4 for a non standard example related to the coboundary condition). The generic constant C below is independent of g and of the parameters n, D, Δ.

Lemma 2.2. *Let n be an integer, and let $\beta \in]0, 1[$, $D > 0$, $\Delta > 0$ be such that $\Delta < \frac{1}{2} n^\beta$. Suppose that $\mathcal{S}(n, D, \Delta)$ holds for the matrices $(M_1, \ldots, M_n)$. Let g be a centered real trigonometric polynomial with $\deg(g) \leq D$. Put $M = \|g\|_\infty$, $q = \|g\|_\infty / \|g\|_2$. Then for $|t| \leq M^{-1} \|S_n\|_2 n^{-\beta}$, the sums $S_n = S_n g$ satisfy:*

$$|\lambda[e^{it\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}t^2}| \leq$$
$$C[\tfrac{M^3|t|^3}{\|S_n\|_2^3} n^{1+2\beta} + \tfrac{M|t|}{\|S_n\|_2} n^{\frac{1+3\beta}{4}} + \tfrac{t^2}{\|S_n\|_2^2} (\|S_n\|_2 n^{\frac{1-\beta}{2}} M\Delta + 2n^{1-\beta} M^2 \Delta^2)]. \quad (20)$$

If $\|S_n\|_2 \geq C\|g\|_2 n^{1/2}$, $2q\Delta \leq n^{\beta/2}$ and $Cq|t| \leq n^{\frac{1-3\beta}{4}}$, we have

$$|\lambda[e^{it\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}t^2}| \leq C[q^3|t|^3 n^{\frac{4\beta-1}{2}} + q|t| n^{\frac{3\beta-1}{4}} + qt^2 \Delta n^{-\frac{\beta}{2}}]. \quad (21)$$

If the decorrelation property (3) holds, then the previous inequality can be replaced by

$$|\lambda[e^{it\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}t^2}| \leq C[q^3|t|^3 n^{\frac{4\beta-1}{2}} + q|t| n^{\frac{3\beta-1}{4}} + q^2 t^2 \Delta^2 n^{-\beta}]. \quad (22)$$

Proof. A) *Replacement of S_n by a sum with "gaps".*

In order to apply Lemma 2.1, we replace the sum S_n by a sum of blocks separated by intervals of length Δ.

For $\beta \in]0, 1[$, D, Δ and g as in the statement of the lemma, we set, for $\leq k \leq u_n - 1$:

$$v_n := \lfloor n^\beta \rfloor, \; u_n := \lfloor n/v_n \rfloor \leq 2n^{1-\beta},$$
$$L_{k,n} := kv_n, \; R_{k,n} := (k+1)v_n - \Delta, \; I_{k,n} := [L_{k,n}, \; R_{k,n}].$$

The sum with "gaps" $S_n'(x)$ is defined by restriction to the intervals $I_{k,n}$:

$$T_{k,n}(x) := \sum_{L_{k,n} < \ell \leq R_{k,n}} g({}^t\!M_\ell \, x), \quad S_n'(x) := \sum_{k=0}^{u_n-1} T_{k,n}(x).$$

The interval $[1, n]$ is divided into u_n blocks of length $v_n - \Delta$ separated by intervals of length Δ. The number of blocks is almost equal to $n^{1-\beta}$ and their length almost equal to n^β. The integers $L_{k,n}$ and $R_{k,n}$ are respectively the left and right ends of the blocks.

Expression of $|T_{k,n}(x)|^2$

$$
\begin{aligned}
|T_{k,n}(x)|^2 &= \Big(\sum_{\ell' \in I_{k,n}} \sum_{p' \in \mathbb{Z}^d} \hat{g}(p')\chi(M_{\ell'}p', x)\Big)\Big(\sum_{\ell \in I_{k,n}} \sum_{p \in \mathbb{Z}^d} \overline{\hat{g}(p)}\chi(-M_\ell\, p, x)\Big) \\
&= \sum_{p,p' \in \mathbb{Z}^d} \sum_{\ell,\ell' \in I_{k,n}} \hat{g}(p')\overline{\hat{g}(p)}\chi(M_{\ell'}p' - M_\ell\, p, x) = \sigma_{k,n}^2 + W_{k,n}(x),
\end{aligned}
$$

with

$$
\begin{aligned}
\sigma_{k,n}^2 &:= \int |T_{k,n}(x)|^2 dx = \sum_{p,p' \in \mathbb{Z}^d} \hat{g}(p')\overline{\hat{g}(p)} \sum_{\ell,\ell' \in I_{k,n}} 1_{M_{\ell'}p' = M_\ell\, p}, \\
W_{k,n}(x) &:= \sum_{p,p' \in \mathbb{Z}^d} \hat{g}(p')\overline{\hat{g}(p)} \sum_{\ell,\ell' \in I_{k,n} : M_{\ell'}p' \neq M_\ell\, p} \chi(M_{\ell'}p' - M_\ell\, p, x).
\end{aligned}
$$

B) *Application of Lemma 2.1.* Now we apply Lemma 2.1 to the array of random variables $(T_{k,n}, 0 \le k \le u_n - 1)$ on the space $(\mathbb{T}^d, \lambda)$. For a fixed n, using the notations of the lemma, we take $u = u_n$, $\zeta_k = T_{k,n}$, for $k = 0, ..., u_n - 1$, (so that $\delta \le Mn^\beta$) and

$$
Y = Y_n = \sum_{k=0}^{u_n - 1} |T_{k,n}|^2;\ a_n = \lambda(Y_n) = \sum_k \sigma_{k,n}^2;\ Q_n(t, x) = \prod_{k=0}^{u_n - 1} (1 + itT_{k,n}(x))\,.
$$

First let us check that $\lambda[Q_n(t, .)] = 1, \forall t$. The expansion of the product gives

$$
Q_n(t, x) = 1 + \sum_{s=1}^{u_n} (it)^s \sum_{0 \le k_1 < \cdots < k_s \le u_n - 1} \prod_{j=1}^s T_{k_j,n}(x).
$$

The products $\prod_{j=1}^s T_{k_j,n}(x)$ are linear combinations of expressions of the type: $\chi(\sum_{j=1}^s M_{\ell_j}p_j, x)$, with $\ell_j \in I_{k_j,n}$ and $\|p_j\| \le D$. So we have $\sum_{j=1}^s M_{\ell_j}p_j \neq 0$ by (8) and therefore $\int \prod_{j=1}^s T_{k_j,n}(x)\ dx = 0$, so that $\int Q_n(t, x)\, dx = 1$. By orthogonality of $(T_{k,n})$, we have also:

$$
\|S_n'\|_2^2 = \lambda(|S_n'|^2) = \lambda(|\sum_{k=0}^{u_n - 1} T_{k,n}|^2) = \sum_{k=0}^{u_n - 1} \lambda(|T_{k,n}|^2) = \sum_{k=0}^{u_n - 1} \sigma_{k,n}^2 = a_n. \tag{23}
$$

B1) *Bounding* $\lambda|Q_n(t, .)|^2$.

$$
\begin{aligned}
\int |Q_n(t, x)|^2\, dx &= \int \prod_{k=0}^{u_n - 1} (1 + t^2 |T_{k,n}(x)|^2)\, dx && (24)\\
&= \int \prod_{k=0}^{u_n - 1} [1 + t^2\sigma_{k,n}^2 + t^2 W_{k,n}(x)]\, dx && (25)\\
&= \prod_{k=0}^{u_n - 1} [1 + t^2\sigma_{k,n}^2] \int \prod_{k=0}^{u_n - 1} [1 + \frac{t^2}{1 + t^2\sigma_{k,n}^2} W_{k,n}(x)]\, dx && (26)
\end{aligned}
$$

The products $W_{k_1}(x)...W_{k_s}(x)$, $0 \le k_1 < ... < k_s < u_n$, are linear combinations of expressions of the form $\chi(\sum_{j=1}^s [M_{\ell_j'}p_j' - M_{\ell_j}p_j], x)$, where $\ell_j, \ell_j' \in I_{k_j,n}$, $M_{\ell_j'}p_j' \neq M_{\ell_j}p_j$, $j = 1, ..., s$, and p_j, p_j' are vectors in $\mathbb{Z}^d$ with norm $\le D$.

As $\mathcal{S}(n, D, \Delta)$ is satisfied, the choice of the gap in the definition of the intervals $I_{k_j,n}$ implies $\sum_{j=1}^s (M_{\ell_j'}p_j' - M_{\ell_j}p_j) \neq 0$ and so the integral of the second factor in

(26) reduces to 1. Now it follows from (26) and the inequality $1 + y \leq e^y, \forall y \geq 0$:

$$e^{-a_n t^2} \int |Q_n(t,x)|^2 \, dx = e^{-a_n t^2} \prod_{k=0}^{u_n-1} [1 + t^2 \sigma_{k,n}^2] \leq e^{-a_n t^2} \, e^{t^2 \sum_{k=0}^{u_n-1} \sigma_{k,n}^2} = 1.$$

B2) *Bound for* S'_n. First we have

$$u_n \delta_n^3 = u_n \max_{0 \leq k \leq u_n - 1} \|T_{k,n}\|_\infty^3 \leq C M^3 n^{1-\beta} \, n^{3\beta} = C M^3 n^{1+2\beta}. \tag{27}$$

Then for $\|Y_n - a_n\|_2$, observe that $\mathbb{E}[(T_{k,n}^2 - \mathbb{E}\, T_{k,n}^2)(T_{k',n}^2 - \mathbb{E}\, T_{k',n}^2)] = 0, \forall 1 \leq k < k' \leq L$, so that the following inequality holds (with $L = u_n$, recall that u_n is of order $n^{1-\beta}$):

$$\begin{aligned} \| \sum_k T_{k,n}^2 - \sum_k \sigma_{k,n}^2 \|_2^2 &= \| \sum_{k=0}^{L-1} T_{k,n}^2 - \sum_{k=0}^{L-1} \mathbb{E}\,[T_{k,n}^2] \|_2^2 \\ &= \sum_{k=1}^{L-1} \mathbb{E}\,[(T_{k,n}^2)^2] - (\sum_{k=0}^{L-1} \mathbb{E}\,[T_{k,n}^2])^2 \leq \sum_{k=0}^{L-1} \mathbb{E}\,[T_{k,n}^4] \leq u_n n^{4\beta} M^4, \end{aligned}$$

hence:

$$|t| \|Y_n - a_n\|_2^{\frac{1}{2}} \leq M |t| n^{\frac{1+3\beta}{4}}. \tag{28}$$

Now we can apply (4) of Lemma 2.1 if $M|t|n^\beta \leq 1$ and $M|t|n^{\frac{1+3\beta}{4}} \leq 1$, which reduces to $M|t| \leq n^{-\frac{1+3\beta}{4}}$. We obtain from (27) and (28):

$$|\lambda[e^{itS'_n}] - e^{-\frac{1}{2}\|S'_n\|_2^2 t^2}| \leq \quad C(M^3 |t|^3 n^{1+2\beta} + M|t| n^{\frac{1+3\beta}{4}}), \tag{29}$$

C) *Bound for* $\|S_n - S'_n\|_2$.

$$\begin{aligned} \|S_n - S'_n\|_2^2 &= \int |\sum_{k=0}^{u_n-1} \sum_{R_{k,n} < \ell \leq L_{k+1,n}} g({}^t\!M_\ell \, x)|^2 \, dx \\ &= \sum_{k=0}^{u_n-1} \int | \sum_{R_{k,n} < \ell \leq L_{k+1,n}} g({}^t\!M_\ell \, x)|^2 \, dx \\ &+2 \sum_{0 < k < k' \leq u_n - 1} \int \sum_{R_{k,n} < \ell \leq L_{k+1,n}} g({}^t\!M_\ell \, x) \sum_{R_{k',n} < \ell' \leq L_{k'+1,n}} g({}^t\!M_{\ell'} \, x) \, dx. \end{aligned}$$

The length of the intervals for the sums in the integrals is Δ. The second sum in the previous expression is 0 by (8), since $n^\beta - \Delta > \Delta$. Each integral in the first sum is bounded by $M^2 \Delta^2$. It implies:

$$\|S_n - S'_n\|_2^2 \leq M^2 \Delta^2 \, u_n \leq 2 n^{1-\beta} M^2 \, \Delta^2. \tag{30}$$

Using the previous inequality we have, since $\int (S_n - S'_n) \, d\lambda = 0$,

$$|\lambda[e^{itS_n} - e^{itS'_n}]| \leq \lambda[|1 - e^{it(S_n - S'_n)}|] \leq C|t|^2 \|S_n - S'_n\|_2^2 \leq C|t|^2 \, M^2 \, \Delta^2 \, n^{1-\beta}. \tag{31}$$

By (30) and the mean value theorem, we have

$$\begin{aligned} |e^{-\frac{1}{2}\|S'_n\|_2^2 t^2} - e^{-\frac{1}{2}\|S_n\|_2^2 t^2}| &\leq \frac{1}{2} t^2 |\|S'_n\|_2^2 - \|S_n\|_2^2| \\ &\leq \frac{1}{2} t^2 (2\|S_n\|_2 + \|S_n - S'_n\|_2) \|S_n - S'_n\|_2 \\ &\leq C t^2 \, (\|S_n\|_2 \, n^{\frac{1-\beta}{2}} \, M \, \Delta + n^{1-\beta} M^2 \, \Delta^2). \end{aligned} \tag{32}$$

D) *Conclusion.* Now if $M|t| \leq n^{-\frac{1+3\beta}{4}}$, we can use the previous bounds (29), (31), (32):

$$\begin{aligned}
& |\lambda[e^{itS_n}] - e^{-\frac{1}{2}\|S_n\|_2^2 t^2}| \\
\leq\ & |\lambda[e^{itS_n}] - \lambda[e^{itS'_n}]| + |\lambda[e^{itS'_n}] - e^{-\frac{1}{2}\|S'_n\|_2^2 t^2}| + |e^{-\frac{1}{2}\|S'_n\|_2^2 t^2} - e^{-\frac{1}{2}\|S_n\|_2^2 t^2}| \\
\leq\ & C[t^2\, M^2 \Delta^2\, n^{1-\beta} + |t|^3 M^3\, n^{1+2\beta} + M|t|\, n^{\frac{1+3\beta}{4}} \\
& \qquad + t^2\,(\|S_n\|_2 n^{\frac{1-\beta}{2}} M\Delta + n^{1-\beta} M^2 \Delta^2)] \\
=\ & C[|t|^3 M^3\, n^{1+2\beta} + M|t|\, n^{\frac{1+3\beta}{4}} + t^2\,(\|S_n\|_2 n^{\frac{1-\beta}{2}} M\Delta + 2n^{1-\beta} M^2 \Delta^2)].
\end{aligned}$$

Then, replacing t by $t\|S_n\|_2^{-1}$, we obtain, if $|t| \leq M^{-1}\|S\|_2 n^{-\beta}$,

$$\begin{aligned}
|\lambda[e^{it\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}t^2}| \quad \leq \quad & C[|t|^3 M^3 \|S_n\|_2^{-3}\, n^{1+2\beta} + |t| M\, \|S_n\|_2^{-1}\, n^{\frac{1+3\beta}{4}} \\
& + t^2 \|S_n\|_2^{-2}\,(\|S_n\|_2\, n^{\frac{1-\beta}{2}} M\Delta + 2n^{1-\beta} M^2 \Delta^2)].
\end{aligned}$$

Suppose now that $\|S_n\|_2 \geq C\|g\|_2 n^{\frac{1}{2}}$ and $2q\Delta \leq n^{\beta/2}$. We obtain, if $Cq|t| \leq n^{\frac{1-3\beta}{4}}$:

$$\begin{aligned}
|\lambda[e^{it\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}t^2}| \quad & \leq \quad C[|t|^3 q^3\, n^{\frac{4\beta-1}{2}} + q|t| n^{\frac{3\beta-1}{4}} + t^2\,(n^{\frac{-\beta}{2}} q\Delta + 2n^{-\beta} q^2 \Delta^2)] \\
& \leq \quad C[q^3 |t|^3\, n^{\frac{4\beta-1}{2}} + q|t| n^{\frac{3\beta-1}{4}} + q t^2 \Delta\, n^{\frac{-\beta}{2}}].
\end{aligned}$$

This finish the proof of the lemma, when the decorrelation property is not assumed.

When the decorrelation property (3) holds, we can write

$$|\|S_n\|_2^2 - \|S'_n\|_2^2| \leq C n^{1-\beta} M^2\, \Delta^2, \tag{33}$$

and the mean value theorem gives by (33),

$$|e^{-\frac{1}{2}\|S'_n\|_2^2 t^2} - e^{-\frac{1}{2}\|S_n\|_2^2 t^2}| \leq \frac{1}{2} t^2 |\|S'_n\|_2^2 - \|S_n\|_2^2| \leq C t^2\, n^{1-\beta} M^2\, \Delta^2. \tag{34}$$

We obtain (22), with the same condition on t. □

Rate of convergence in the CLT

The inequalities (21) or (22) and the inequality of Esseen give a way to obtain a rate of convergence in the CLT.

Recall that if X, Y are two real random variables defined on the same probability space with probability $\mathbb{Q}$, their mutual distance in distribution is defined by:

$$d(X, Y) = \sup_{x\in\mathbb{R}} |\mathbb{Q}(X \leq x) - \mathbb{Q}(Y \leq x)|.$$

Let $H_{X,Y}(t) := |\mathbb{E}_{\mathbb{Q}}(e^{itX}) - \mathbb{E}_{\mathbb{Q}}(e^{itY})|$. Take as Y a r.v. with a normal law $\mathcal{N}(0, \sigma^2)$. The inequality of Esseen (cf. [10] p. 512) tells us that if X has a vanishing expectation and if the difference of the distributions of X and Y vanishes at $\pm\infty$, then for every $U > 0$,

$$d(X, Y) \leq \frac{1}{\pi} \int_{-U}^{U} H_{X,Y}(x) \frac{dx}{x} + \frac{24}{\pi} \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{U}.$$

Let Y_1 be a r.v. with standard normal law. Suppose, for instance, that (21) holds with a fixed gap Δ and that $\|S_n\|_2 > Cn^{\frac{1}{2}}$. Taking $X = S_n/\|S_n\|_2$, we have $|H_{X,Y_1}(t)| \leq C \sum_{i=1}^4 n^{-\gamma_i} |t|^{\alpha_i}$, where the constants are given by (21). Thus $d(X, Y_1)$ is bounded by

$$\frac{C}{U} + C \sum_{i=1}^4 n^{-\gamma_i} \frac{1}{\alpha_i} U^{\alpha_i}.$$

In order to optimize the choice of $U = U_n$, we take $U_n = n^\gamma$ with $\gamma = \min_i \frac{\gamma_i}{\alpha_i+1}$. This gives the bound $d(S_n/\|S_n\|_2, Y_1) \leq C n^{-\gamma}$. Then we have to chose the parameter $\beta \in]0, 1[$.

Theorem 2.3. *Let $(A_k)_{k\geq 1}$ be a sequence of matrices taking values in a set $\mathcal{A}$ of matrices in $\mathcal{M}_d^*(\mathbb{Z})$ such that Condition 2 holds. Let $f \in \mathcal{H}_0(\mathbb{T}^d)$ be such that $\|S_n f\| \geq C n^{\frac{1}{2}}$, for a constant $C > 0$, for n big enough. Then $S_n f$ satisfies the CLT with a rate $d(\frac{S_n f}{\|S_n f\|_2}, Y_1) = O(n^{-\rho})$, for every $\rho < 1/32$ (for every $\rho < 1/20$ with the decorrelation property (3)).*

Proof. Proposition 1 shows that there exist an integer L and a uniformly bounded sequence (g_n) of trigonometric polynomials such that $\|g_n\|_\infty \leq \|f\|_\infty$, $\deg(g_n) \leq n^L$ and $\|S_n f - S_n g_n\|_2 \leq n^{-4}$. For n big enough, $\|S_n g_n\|_2 > \frac{1}{2} C n^{\frac{1}{2}}$. This implies:

$$\begin{aligned} & |\lambda[e^{it\frac{S_n f}{\|S_n f\|_2}}] - \lambda[e^{it\frac{S_n g_n}{\|S_n g_n\|_2}}]| \\ \leq \quad & C|t|^2 \lambda(|\frac{S_n f}{\|S_n f\|_2} - \frac{S_n g_n}{\|S_n g_n\|_2}|^2) \leq C t^2 \frac{\|S_n f - S_n g_n\|_2^2}{\|S_n f\|_2 \|S_n g_n\|_2} \leq C t^2 n^{-9}. \end{aligned}$$

By Proposition 4, Property $\mathcal{S}(n, n^4, 4C_S \ln n)$ holds and we can apply Lemma 2.2 to the trigonometric polynomial g_n. We obtain:

$$|\lambda[e^{it\frac{S_n f}{\|S_n f\|_2}}] - e^{-\frac{1}{2}t^2}| \leq C[|t|^3 n^{\frac{4\beta-1}{2}} + |t| n^{\frac{3\beta-1}{4}} + |t|^2 (\ln n)^2 n^{-\frac{\beta}{6}} + t^2 n^{-9}].$$

Now we apply the method recalled a few lines above. When (21) holds, we compute $\min(\frac{1-4\beta}{8}, \frac{1-3\beta}{8}, \frac{\beta-\varepsilon}{6}, 3)$, for $\varepsilon > 0$ small. Choosing $\beta = \frac{3}{16} + \frac{\varepsilon}{4}$, we obtain the rate of convergence $\frac{1}{32} - \frac{\varepsilon}{8}$.

With the decorrelation property (22), we compute $\min(\frac{1-4\beta}{8}, \frac{1-3\beta}{8}, \frac{\beta-\varepsilon}{3}, 3)$. Taking $\beta = \frac{3}{20} + \frac{2}{5}\varepsilon$, we obtain the rate of convergence $\frac{1}{20} - \frac{\varepsilon}{5}$. □

Application to superlacunary sequences

In dimension 1 the superlacunary growth condition of (q_n) is a sufficient condition (Salem-Zygmund) for the CLT and this extends to $d > 1$.

Theorem 2.4. *Let $f \in \mathcal{H}_0$. Under Condition 4 5 the asymptotic variance $\sigma^2(f) = \lim_n \frac{1}{n}\|S_n f\|_2^2$ exists, $\sigma^2(f) = \|f\|_2$ and the CLT holds if $\|f\|_2 \neq 0$.*

Proof. By Proposition 2, Condition 5 (which follows from Condition 4) implies $\lim_n \frac{1}{n}\|S_n f\|_2^2 = \|f\|_2$. By Proposition 5 the separation of frequencies is satisfied. We conclude as in the previous theorem. □

2.4. **A non standard example.** Now we illustrate the questions of variance and coboundary by a simple example. Let A, B be two matrices in $\mathrm{SL}(2, \mathbb{Z})$ with positive coefficients (hence the corresponding automorphism $\tau_A : x \mapsto Ax \mod 1$ is ergodic on $(\mathbb{T}^2, \lambda)$) such that AB^{-1} is hyperbolic. Let $J = (n_k)$ be an increasing sequence and let $(A_j)_{j\geq 1}$ be the sequence of matrices defined by

$$A_j = A \text{ if } j \notin J, \quad = B \text{ if } j \in J.$$

The following proposition shows how the behavior of the sums $\sum_{k=1}^n f(A_k \ldots A_1 x)$ can depend on the coboundary condition.

Proposition 6. *Let f be a non zero function in $\mathcal{H}_0(\mathbb{T}^2)$, $(n_k)_{k\geq 1} = (\lfloor k^L \rfloor)_{k\geq 1}$, for $L \geq 1$. If f is not a coboundary for τ_A, then we have the convergence in distribution with a constant $\sigma > 0$:*

$$\frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} f(A_k \dots A_1 x) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

If $f = h - h(A\cdot)$ then, $g(\cdot) = h(\cdot) - h(AB^{-1}\cdot)$ is not zero and

$$\frac{1}{\|g\|_2}\, n^{-\frac{1}{2L}} \sum_{k=1}^{n} f(A_k \dots A_1 \cdot) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

Proof. 1) Let us compare the variance of $S_n f(x) = \sum_{k=1}^n f(A_k \dots A_1 x)$ with the variance of the ergodic sums associated to the action of A. We have

$$\mathbb{E}((S_n f(x))^2) = n \int_{\mathbb{T}^d} f^2(x)dx + 2\sum_{k<\ell} \langle f(A_1^\ell \cdot), f(A_1^k \cdot)\rangle.$$

Condition 3 is satisfied by matrices with positive coefficients (see subsection 3.2), hence by (15) Proposition 1.9, for $\kappa < 1$: $\langle f(A_1^\ell \cdot), f(A_1^k \cdot)\rangle = \langle f(A_{k+1}^\ell \cdot), f(\cdot)\rangle \leq C\kappa^{\ell-k}$, so that

$$|\mathbb{E}(S_n f(x))^2 - [n \int_{\mathbb{T}^d} f^2\, dx + 2 \sum_{k<\ell \text{ and } \ell-k\leq n^\alpha} \langle f(A_1^\ell \cdot), f(A_1^k \cdot)\rangle]| \leq C\kappa^{n^\alpha} n^2.$$

Let $r_n := \#(J \cap [1,n])$. If k is at distance $\geq n^\alpha$ from J and $\ell - k \leq n^\alpha$, then $A_{k+1}^\ell = A^{\ell-k}$. The number of blocks A_{k+1}^ℓ containing B with $k < \ell$ and $\ell - k \leq n^\alpha$ is less than $r_n n^{2\alpha}$. Thus we have

$$|\mathbb{E}((S_n f(x))^2) - [n \int_{\mathbb{T}^d} f^2\, dx + 2 \sum_{k<\ell, \ell-k\leq n^\alpha} \langle f(A^\ell \cdot), f(A^k \cdot)\rangle]| \leq C(\kappa^{n^\alpha} n^2 + n^{2\alpha} r_n),$$

and if $n^{2\alpha-1} r_n$ tends to 0, then $\frac{1}{n}\mathbb{E}((S_n f(x)^2) - \frac{1}{n}\mathbb{E}((\sum_{k=1}^n f(A^k x))^2) \to 0$.

If $n_k = \lfloor k^L \rfloor$ for $L > 1$, then r_n is equivalent to $n^{1/L}$. Taking $\alpha < \frac{1}{2}(1 - L^{-1})$ we get

$$\lim \frac{1}{n}\mathbb{E}((\sum_{k=1}^{n} f(A_k \dots A_1 x))^2) = \sigma^2 := \int_{\mathbb{T}^d} f(x)dx + 2\sum_{k=1}^{\infty} \langle f, f(A^k \cdot)\rangle.$$

Thus, if f is not a coboundary for the action of A, we have $\|S_n f\| \geq C n^{\frac{1}{2}}$ for some $C > 0$. Moreover, as A and B have positive coefficients, Corollary 2 in section 3 ensures that Condition 2 holds for products of matrices A and B. This implies that Theorem 2.3 applies. In particular we have $\frac{1}{\sigma\sqrt{n}} \sum_{k=1}^n f(A_k \dots A_1 x) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

2) Now let us assume that f is a coboundary for A: $f(\cdot) = h(\cdot) - h(A\cdot)$. As f is Hölder continuous, it is known that h is also Hölder continuous. Putting $n_0 = 0$,

we can rearrange the sums $S_n f(x) = \sum_{k=1}^n f(A_k \dots A_1 x)$:

$$\begin{aligned} S_n f(x)) &= \sum_{k=1}^{r_n} \sum_{j=n_{k-1}}^{n_k-1} f(A_j \dots A_1 x) + \sum_{j=n_{r_n}}^{n} f(A_j \dots A_1 x) - f(x) \\ &= \sum_{k=1}^{r_n} \left(\sum_{j=n_{k-1}}^{n_k-1} h(A_j \dots A_1 x) - \sum_{j=n_{k-1}}^{n_k-1} h(AA_j \dots A_1 x) \right) \\ &\qquad + \sum_{j=n_{r_n}}^{n} h(A_j \dots A_1 x) - \sum_{j=n_{r_n}}^{n} h(AA_j \dots A_1 x) - f(x). \end{aligned}$$

For $k = 1, \dots, r_n$, $j = n_{k-1}, \dots, n_k - 2$ and $j = n_{r_n}, \dots, n-1$, we have the equality $h(AA_j \dots A_1 x) = h(A_{j+1} A_j \dots A_1 x)$. Thus

$$\begin{aligned} S_n f(x) &= \sum_{k=1}^{r_n} [h(A_{n_{k-1}} \dots A_1 x) - h(AA_{n_k-1} \dots A_1 x)] \\ &\qquad + h(A_{n_{r_n}} \dots A_1 x) - h(AA_n \dots A_1 x) - f(x). \end{aligned}$$

Define $g(\cdot) = h(\cdot) - h(AB^{-1} \cdot)$ and $M_j = BA^{n_j - n_{j-1} - 1}$. The function g is not $\equiv 0$ because otherwise, since AB^{-1} is ergodic, h would be constant and $f \equiv 0$. As $f(x) = h(x) - h(Ax)$ and $A_{n_k} \dots A_1 = BA_{n_k-1} \dots A_1$, we can rearrange the expression above and get

$$\begin{aligned} S_n f(x) &= \sum_{k=1}^{r_n} [h(BA_{n_k-1} \dots A_1 x) - h(AA_{n_k-1} \dots A_1 x)] \\ &\qquad + h(Ax) - h(AA_n \dots A_1 x) \\ &= \sum_{k=1}^{r_n} g(M_k \dots M_1 x) + h(Ax) - h(AA_n \dots A_1 x). \end{aligned}$$

For the asymptotic variance of the sums associated to the sequence (M_j), we have :

$$\begin{aligned} \mathbb{E}((\sum_{k=1}^{n} g(M_k \dots M_1 x))^2) - n \int_{\mathbb{T}^d} g^2(x)dx &= 2 \sum_{k<\ell} \langle g(M_1^\ell \cdot), g(M_1^k \cdot) \rangle \\ &= 2 \sum_{k<\ell} \langle g(M_{k+1}^\ell \cdot), g(\cdot) \rangle. \end{aligned}$$

When $n_k = \lfloor k^L \rfloor$ with $L > 1$, M_{k+1}^ℓ is a product of $\lfloor \ell^L \rfloor - \lfloor k^L \rfloor$ matrices A or B, we have $|\langle g(M_{k+1}^\ell \cdot), g(\cdot) \rangle| \leq C \kappa^{\ell^L - k^L}$, so that

$$|\sum_{k<\ell} \langle g(M_{k+1}^\ell \cdot), g(\cdot) \rangle| \leq C \sum_{k=1}^{n-1} \sum_{\ell=k+1}^{n} \kappa^{\ell^L - k^L} \leq C \sum_{k=1}^{n-1} \kappa^{L k^{L-1}} < \infty,$$

and $\lim_n \frac{1}{n} \mathbb{E}((\sum_{k=1}^n g(M_k \dots M_1 x))^2) = \int_{\mathbb{T}^d} g^2(x)dx$. When $n_k = \lfloor k^L \rfloor$ with $L > 1$, Condition 4 (hence Condition 5) holds for M_1^n and Theorem 2.4 implies the convergence in distribution

$$\frac{1}{\|g\|_2 \sqrt{r_n}} \sum_{k=1}^{r_n} g(M_k \dots M_1 \cdot) \to \mathcal{N}(0,1).$$

This proves the second assertion of the proposition. $\square$

3. Stationary products, examples.

In this section we consider a sequence (M_k) in $\mathcal{M}_d^*(\mathbb{Z})$ and the sums $S_n f(x) = \sum_{k=1}^n f({}^t\!M_k x)$. When the matrices M_k are positive, this can be viewed as a generalization of the trigonometric sums $S_n f(x) = \sum_{k=1}^n f(q_k x)$ (cf. references in the introduction).

Remark that there are examples of sequences (M_k) of positive matrices with an exponential growth for which the convergence in law to a standard normal law does not hold. This is the case in dimension 1 with the sequence $q_n = 2^n - 1, n \geq 1$, an example due to Fortet and Erdös, and in higher dimension examples can be constructed (see [7]).

In the study of the behavior of the sums $S_n f$, the following questions arise:
- decorrelation property of the sequence $(f({}^t\!M_k x))_{k\geq 1}$, for a control on the variance,
- non nullity of the variance for the non degeneracy of the limit.

The latter question seems to be out of reach outside the superlacunary case, even for arithmetic sums in dimension 1, where generally non degeneracy is assumed, but difficult or impossible to check. The reason is that in general, given a sequence (M_n) and a regular or polynomial function f, it is difficult to know if the variance is non zero, even in dimension 1 for a sequence of (q_n) and even when the sequence is obtained as a product.

Nevertheless, the situation is much better in the stationary case, when the sequence (M_n) is obtained as a product of stationary matrices, or integers. Then some information can be obtained on the non nullity of the variance.

A special case is when the matrices $A_k(\omega)$ are chosen at random and independently (see [3] for toral automorphisms). The case of $\mathrm{SL}(d, \mathbb{Z})$ extends to the following general setting: let G be a group of measure preserving transformations on a probability space (X, λ) and let μ be a probability measure on G. If a spectral gap is available for the convolution by μ on $L_0^2(X \times X, \lambda \otimes \lambda)$, then a "quenched" CLT for functions f in $L_0^p(\mu)$, $p > 2$, can be shown (cf. [8]). Moreover the spectral gap implies the non degeneracy of the CLT. Therefore we will not consider here specifically the independent case, but nevertheless remark that the method which is used here applies to the independent case for the action of matrices in $\mathrm{SL}(d, \mathbb{Z})$ on the torus and a CLT with rate for Hölder functions can be obtained in this way. We consider here different situations where a CLT can be proved for *stationary*, not necessarily independent, sequences of automorphisms.

3.1. Stationary products. Ergodicity, variance

In this section we consider a stationary process $(A_k(\omega))$ with values in $\mathcal{M}_d^*(\mathbb{Z})$ and the corresponding products ${}^t\!M_k = {}^t\!A_k(\omega)...{}^t\!A_1(\omega)$. Stationarity can be expressed via a measure preserving transformation.

Let θ be an invertible measure preserving ergodic transformation on a probability space (Ω, μ). Let $\mathcal{A}$ be a set of matrices in $\mathcal{M}_d^*(\mathbb{Z})$.

Notation. Let $\omega \to A(\omega)$ be a measurable map from Ω to $\mathcal{A}$. Let τ be the map $\omega \to \tau(\omega)$ from Ω to the semigroup of endomorphisms of $\mathbb{T}^d$ where $\tau(\omega)\, x = {}^t\!A(\omega)\, x$. We define the skew product θ_τ on the product space $\Omega \times \mathbb{T}^d$ equipped with the product measure $\nu := \mu \otimes \lambda$ by $\theta_\tau : (\omega, x) \mapsto (\theta\omega, \tau(\omega)x)$.

For $\omega \in \Omega$ and f a function on $\mathbb{T}^d$, we write $S_n(\omega, f)(x) = \sum_{k=1}^n f({}^t\!A_1^k(\omega)x)$, where

$$A_i^j(\omega) = A(\theta^{i-1}\omega)A(\theta^i\omega)...A(\theta^{j-1}\omega), \ j \geq i \geq 1.$$

In this framework we use the following versions of Conditions 1 and 2 of Subsection 2.2.

Condition 6. *There is $C_1 > 0$ such that for a.e. ω, for every $p, q \in \mathbb{Z}^d \setminus \{0\}$ with $\|p\|, \|q\| \leq D$,*

$$A_1^r(\theta^\ell \omega)p \neq q, \ \forall r > C_1 \log D, \ \forall \ell \geq 0. \tag{35}$$

Condition 7. *For a.e. ω, there exist $\gamma > 1$, c and $C > 0$ such that for every $p \in \mathbb{Z}^d \setminus \{0\}$*

$$\|A_1^{\ell+r}(\omega)p\| \ \geq c\gamma^{r - C\log\|p\|}\|A_1^\ell(\omega)\|, \ \forall r > C_1 \log\|p\|, \ \forall \ell \geq 1. \tag{36}$$

Proposition 7. *Assume Condition 6. If (Ω, μ, θ) is ergodic, then the dynamical system $(\Omega \times \mathbb{T}^d, \theta_\tau, \mu \otimes \lambda)$ is ergodic. The system $(\Omega \times \mathbb{T}^d, \theta_\tau, \mu \otimes \lambda)$ is mixing on the functions f in $L^2(\Omega \times \mathbb{T}^d)$ which are orthogonal to the subspace of functions depending only on ω. For $f \in \mathcal{H}_0(\mathbb{T}^d)$ the decorrelation holds with an exponential rate and the variance exists.*

Proof. Let $g \in L^2(\Omega \times \mathbb{T}^d)$ be a trigonometric polynomial with respect to x of degree D, orthogonal to functions depending only on ω, $g(\omega, x) = \sum_{0<\|p\|\leq D} g_p(\omega)\chi(p, x)$. We have:

$$\begin{aligned}\langle g \circ \theta_\tau^n, g\rangle_\nu &= \sum_{p,q} \int\int g_p(\theta^n\omega)\chi(A_1^n(\omega)p, x)\,\overline{g_q(\omega)\chi(q,x)}\; dx\; d\mu(\omega) \\ &= \sum_{p,q} \int g_p(\theta^n\omega)\,\overline{g_q(\omega)}\; 1_{A_1^n(\omega)p=q}\; d\mu(\omega).\end{aligned}$$

Condition 6 (with $\ell = 0$) implies that there is a constant C_1 not depending on D such that $A_1^n(\omega)p \neq q$, for $n \geq C_1 \ln D$. Thus we have $\langle g \circ \theta_\tau^n, g\rangle = 0$, for $n \geq C_1 \ln D$.

With a density argument this shows that $\lim_n \langle g \circ \theta_\tau^n, g\rangle_\nu = 0$ for all functions g in $L^2(\nu)$ which are orthogonal to functions depending only on ω. If the system (Ω, μ, θ) is ergodic, this implies ergodicity of the extension.

For $f \in \mathcal{H}_0(\mathbb{T}^d)$, using the approximation argument as in Proposition 2, we obtain: $|\langle f \circ \theta_\tau^n, f\rangle| = O(\kappa^n)$, for a constant $\kappa \in [0, 1[$. It is well known that the summability of the series of decorrelations implies the existence of the asymptotic variance. Therefore, we have $\frac{1}{n}\|S_n f\|_{2,\nu}^2 \to \sigma^2(f)$. ☐

We are going to prove that, for a.e. ω, $\lim_n n^{-1}\|S_n(\omega, f)\|_2^2 = \sigma^2(f)$. Hence the limit does not depend on ω. Here the norm $\|S_n(\omega, f)\|_2$ is taken *with respect to x and ω is fixe*d.

Proposition 8. *Assume Condition 6. For every $f \in \mathcal{H}_0(\mathbb{T}^d)$, for μ-a.e. $\omega \in \Omega$, the sequence $(n^{-\frac{1}{2}}\|S_n(\omega, f)\|_2)$ converges to the variance $\sigma(f)^2 = \lim_n \frac{1}{n}\|S_n f\|_{2,\nu}^2$ given by the skew product. Moreover $\sigma(f) = 0$ if and only if f satisfies in L^2 the coboundary condition: there exists $g \in L^2(\nu)$ such that*

$$f(x) = g(\omega, x) - g(\theta\omega, \tau(\omega)x), \ \nu - \text{a.e.} \tag{37}$$

Proof. The system $(\Omega \times \mathbb{T}^d, \theta_\tau, \mu \times dx)$ is ergodic according to Proposition 7. We have $S_n(\omega, f)(x) = \sum_{k=0}^{n-1} F(\theta_\tau^k(\omega, x))$, where $F(\omega, x) := f(x)$. Hence:

$$\begin{aligned}
\frac{1}{n}\|S_n(\omega, f)\|_2^2 &= \frac{1}{n}\sum_{\ell=0}^{n-1}\sum_{\ell'=0}^{n-1}\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, x))F(\theta_\tau^{\ell'}(\omega, x))\,dx \\
&= \|f\|^2 + \frac{2}{n}\sum_{r=1}^{n-1}\sum_{\ell=0}^{n-1}\int_{\mathbb{T}^d}(F.F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega, x))\,dx \\
&- \frac{2}{n}\sum_{r=1}^{n-1}\sum_{\ell=n-r}^{n-1}\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, x))F(\theta_\tau^{\ell+r}(\omega, x))\,dx.
\end{aligned}$$

By Condition 6 (hence Condition 1), we can apply (15) of Proposition 2. For a constant $C(f)$ and a real $\kappa < 1$, we have: $|\int_{\mathbb{T}^d} f(x)\, f(A(\theta^{\ell+r}\omega)...A(\theta^{\ell+1}\omega)x)\,dx| \le C(f)\,\kappa^r$; hence

$$\left|\frac{1}{n}\sum_{r=1}^{n-1}\sum_{\ell=n-r}^{n-1}\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, x))\,F(\theta_\tau^{\ell+r}(\omega, x))\,dx\right| \quad\le\quad C(f)\,\frac{1}{n}\sum_{r=1}^{n-1} r\kappa^r \to 0$$

and the convergence of $\frac{1}{n}\|S_n(\omega, f)\|_2^2$ reduces to that of

$$\|f\|^2 + \frac{2}{n}\sum_{r=1}^{n-1}\sum_{\ell=0}^{n-1}\int_{\mathbb{T}^d}(F.F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega, x))\,dx. \tag{38}$$

For μ-a.e. ω, for every r, by the ergodic theorem

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{n}\sum_{\ell=0}^{n-1}\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, x))F(\theta_\tau^{\ell+r}(\omega, x))\,dx \\
= \;&\lim_n \frac{1}{n}\sum_{\ell=0}^{n-1}\int_{\mathbb{T}^d} f(x)\;f(A(\theta^{\ell+r}\omega)...A(\theta^{\ell+1}\omega)x)\,dx = \int_{\Omega\times\mathbb{T}^d}(F.F\circ\theta_\tau^r)\,d\omega\,dx.
\end{aligned}$$

In view of (15), $\frac{1}{n}|\sum_{\ell=0}^{n-1}\int_{\mathbb{T}^d} f(x)\;f(A(\theta^{\ell+r}\omega)...A(\theta^{\ell+1}\omega)x)\,dx|$ is bounded, uniformly with respect to n by the general term of a converging series. Therefore we can take the limit for μ-a.e. ω in (38):

$$\lim_{n\to+\infty}\frac{1}{n}\|S_n(\omega, f)\|^2 = \|f\|_2^2 + 2\sum_{r=1}^{+\infty}\int_{\Omega\times\mathbb{T}^d}(F.F\circ\theta_\tau^r)\,d\omega\,dx = \sigma^2(f).$$

It is known that, in the case of summable decorrelations, $\sigma = 0$ if and only if f is a coboundary, i.e. satisfies (37), with g square integrable. $\square$

Remark 2. 1) The previous proof shows that for a uniquely ergodic system (Ω, μ, θ) defined on a compact space Ω (for instance an ergodic rotation on a torus), the convergence of the variance given in Proposition 8 holds for every $\omega \in \Omega$, if the map τ is continuous outside a set of μ-measure 0.

2) It can be shown that there is a set $\Omega_1 \subset \Omega$ of full measure such that, for $\omega \in \Omega_1$, convergence in Proposition 8 holds for every $f \in \mathcal{H}^0(\mathbb{T}^d)$.

3) If (Ω, μ, θ) has no eigenvalues, then by Proposition 7 $(\Omega \times \mathbb{T}^d, \mu \times \lambda, \theta_\tau)$ has also a continuous spectrum. If we take $f = 1_E - \lambda(E)$ where E is a Borel set with $0 < \lambda(E) < 1$, then f is not a coboundary. Indeed if f is coboundary in the skew

product, $f = \varphi - \varphi \circ \theta_\tau$, then $e^{2\pi i \varphi \circ \theta_\tau} = e^{2\pi i f}\, e^{2\pi i \varphi} = e^{-2\pi i \lambda(E)}\, e^{2\pi i \varphi}$. This implies that $e^{2\pi i \varphi}$ is a constant and $\lambda(E) \in \mathbb{Z}$, a contradiction.

Non nullity of the variance

Now we assume Condition 6 and we consider the condition of coboundary

$$f(x) = g(\omega, x) - g(\theta\omega, \tau(\omega)x)$$

In this paragraph we make remarks on the coboundary condition. In 3.4 we will show that for $d = 1$ the coboundary obstruction to the CLT never occurs.

For $j, p \in \mathbb{Z}^d$, we denote by $D(j, p, \omega)$ the set $\{k \geq 0 : A_0^k(\omega)j = p\}$ and by $c(j, p, \omega)$ its cardinality. Condition 6 implies the following fact:

Lemma 3.1. $\sup_{j \in J,\, p \in \mathbb{Z}^d}\ c(j, p, \omega) < \infty$, *for every finite subset J of $\mathbb{Z}^d \setminus \{0\}$.*

Proof. Let j be in J and let $k_1 := \inf_{k \in D(j,p,\omega)} k$, if the set is non void. If k_2 belongs to $D(j, p, \omega)$ with $k_2 > k_1$, then $A_0^{k_2}(\omega)j = p = A_0^{k_1}(\omega)j$, so that $A_0^{k_1}(\omega)(A_{k_1+1}^{k_2}(\omega)j - j) = 0$, hence $A_{k_1+1}^{k_2}(\omega)j = j$. According to (35), this implies that the number of such integers k_2 is finite and bounded independently of p. As J is finite, the result follows. □

Proposition 9. *Let f be a trigonometric polynomial. If there exists $g \in L^2(\Omega \times \mathbb{T}^d)$ such that $f = g - g \circ \theta_\tau$, then g is also a trigonometric polynomial with respect to x.*

Proof. Let $f = \sum_{j \in J} f_j \chi_j$, where J is a finite subset of $\mathbb{Z}^d$. Let g be in L^2 such that $f(x) = g(\omega, x) - g(\theta\omega, \tau(\omega)x)$. This coboundary relation gives $f \circ \theta_\tau^k = g \circ \theta_\tau^k - g \circ \theta_\tau^{k+1}$, then $\sum_{n=1}^N \sum_{k=0}^{n-1} f \circ \theta_\tau^k = Ng - \sum_{n=1}^N g \circ \theta_\tau^n$; hence

$$\begin{aligned} g - \frac{1}{N} \sum_1^N g \circ \theta_\tau^k &= \sum_{k=0}^{N-1} (1 - \frac{k}{N}) f \circ \theta_\tau^k \\ &= \sum_{p \in \mathbb{Z}^d} \sum_{k=0}^{N} [\sum_{j\,:\, A_0^k(\omega)j = p} (1 - \frac{k}{N})\ f_j]\, \chi_p. \end{aligned} \tag{39}$$

As g belongs to L^2, by ergodicity we deduce the convergence in L^2-norm

$$g = \lim_N \sum_{k=0}^{N-1} (1 - \frac{k}{N}) f \circ \theta_\tau^k.$$

Moreover the maximal function $G := \sup_N \frac{1}{N} |\sum_1^N g \circ \theta_\tau^k|$ is square integrable and

$$\sup_N |\sum_{k=0}^{N-1} (1 - \frac{k}{N}) f \circ \theta_\tau^k|^2 \leq |g|^2 + 2|g||G| + |G|^2 \in L^2(\mu),$$

hence, for a.e. ω, there is $M(\omega) < \infty$ such that

$$\sup_N \sum_{p \in \mathbb{Z}^d} |\sum_{k=0}^{N-1} [\sum_{j\,:\, A_0^k(\omega)j = p} (1 - \frac{k}{N})\ f_j]|^2 < M(\omega). \tag{40}$$

If N goes to ∞, the expression $\sum_{k=0}^{N-1} [\sum_{j\,:\, A_0^k(\omega)j = p} (1 - \frac{k}{N})\ f_j]$ tends to the finite sum $\sum_{j \in J} c(j, p, \omega)\ f_j$ (cf. Lemma 3.1). By restricting first the sums in

(40) to a finite set of indices p and passing to the limit with respect to N in $\sum_p |\sum_{k=0}^{N-1} [\sum_{j\,:\,A_0^k(\omega)j=p}(1-\frac{k}{N})\ f_j]|^2$, we obtain

$$\sum_{p\in\mathbb{Z}^d} |\sum_{j\in J} c(j,p,\omega)\ f_j|^2 < M(\omega). \tag{41}$$

According to Lemma 3.1, for every ω, $c(j,p,\omega)$ takes only a finite number of values, when j belongs to the finite fixed set J and p to $\mathbb{Z}^d$. Therefore the sequence $(|\sum_{j\in J} c(j,p,\omega)f_j|)_{p\in\mathbb{Z}^d}$ takes only a finite number of distinct non zero values. Let $\delta > 0$ be a lower bound of these values.

Inequality (41) then yields $\delta^2\ \#\{p\in\mathbb{Z}^d : \sum_{j\in J} c(j,p,\omega)f_j \neq 0\} \leq M(\omega)$, so that the cardinal is finite for a.e. ω. This shows that g is a trigonometric polynomial. □

Corollary 1. *If f is a coboundary and has non negative Fourier coefficients, then $f(x) = 0$ a.e.*

Proof. Using the fact that $c(j,p,\omega)\in\mathbb{N}$, we obtain:

$$\|g(\omega,.)\|_2^2 = \sum_p (\sum_{j\in J} c(j,p,\omega)f_j)^2 \geq \sum_p (\sum_{j\in J} c(j,p,\omega)f_j^2) \geq \sum_{j\in J}(\sum_p c(j,p,\omega))f_j^2.$$

For $j\neq 0$, we have $\sum_p c(j,p,\omega) = +\infty$, thus $f_j = 0$ for $j \neq 0$, and $f_0 = 0$ because f is a coboundary. □

In both examples given below, Condition 1 is satisfied and therefore the results on coboundaries apply.

3.2. **Example 1: 2x2 positive matrices.** The first example for which we obtain a CLT for a.e. ω in the stationary case is that of positive matrices in $\mathrm{SL}(2,\mathbb{Z})$. The computation in this case is elementary.

We consider a finite set $\mathcal{A}$ of matrices in $\mathrm{SL}(2,\mathbb{Z}_+)$ with positive coefficients. We study the asymptotical behavior of the products $A_i^j := A_i...A_j$, where $A_i,...,A_j$, $i\leq j$, is any choice of matrices in $\mathcal{A}$.

Let M be a 2×2 matrix with >0 coefficients with real eigenvalues $r = r(M)$, $s = s(M)$, $r > s$. Let $F = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be such that

$$M = F\begin{pmatrix} r & 0 \\ 0 & s\end{pmatrix}F^{-1} = \begin{pmatrix}(r-s)u+s & -(r-s)v \\ (r-s)w & -(r-s)u+r\end{pmatrix},$$

with $ad-bc = 1$.

We have $M = \begin{pmatrix}(r-s)u+s & -(r-s)v \\ (r-s)w & -(r-s)u+r\end{pmatrix}$, with $u = ad \in]0,1[$, $v = ab < 0$, $w = cd > 0$, since the positivity of the coefficients of M implies that $v<0$, $w>0$, and, by multiplying the relation $ad-bc = 1$ by ad, it follows $u^2 - vw = u$, thus $u^2 - u = vw < 0$.

Lemma 3.2. *There exist a constant c such that for every p in $\mathbb{Z}^2\setminus\{0\}$ and every product M of n matrices taking values in $\mathcal{A}$, if $n\geq c\ln\|p\|$, then $Mp\in\mathbb{R}_+^2\cup\mathbb{R}_-^2$.*

Proof. Let $\lambda := \frac{w}{u} = \frac{u-1}{v}$. We have $\lambda > 0$ and we can rewrite the matrix M as

$$M = r\begin{pmatrix} u & \lambda^{-1}(1-u) \\ \lambda u & 1-u\end{pmatrix} + s\begin{pmatrix} 1-u & -\lambda^{-1}(1-u) \\ -\lambda u & u\end{pmatrix}.$$

Thus, for every vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $Mx = r(ux + \lambda^{-1}(1-u)y)\begin{pmatrix} 1 \\ \lambda \end{pmatrix} + s(x - \lambda^{-1}y)\begin{pmatrix} 1-u \\ -\lambda u \end{pmatrix}$.

The eigenvectors of M are $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ and $\begin{pmatrix} 1-u \\ -\lambda u \end{pmatrix}$, corresponding respectively to the eigenvalues r and s.

As M is a product of n matrices of $\mathcal{A}$, it maps the cone $\mathbb{R}^2_+$ strictly into itself: $M\mathbb{R}^2_+ \subset \bigcup_{A\in\mathcal{A}} A\mathbb{R}^2_+$. It follows that the slope $\lambda = \lambda(M)$ of the positive eigenvector of M is bounded from below and above by constants which only depend on $\mathcal{A}$: there exists $\delta > 0$ such that $\delta \le \lambda \le \delta^{-1}$.

Let us write $r\zeta + \varphi$ and $\lambda r\zeta + \psi$ the components of MX with:

$$\zeta := ux + \lambda^{-1}(1-u)y, \ \varphi := s(x - \lambda^{-1}y)(1-u), \ \psi := -s(x - \lambda^{-1}y)\lambda u.$$

There exist constants $C' > 0$ and $\gamma > 1$ such that the positive eigenvalue $r(M)$, for M a product of n matrices taking values in $\mathcal{A}$, satisfies: $r(M) \ge C'\gamma^n$.

As $s(M) = r(M)^{-1}$, we have $s(M) \le C'^{-1}\gamma^{-n}$ and, as $\delta \le \lambda \le \delta^{-1}$,

$$\max(|\varphi|, |\psi|) \le C'^{-1}\delta^{-1}\gamma^{-n}\|X\|.$$

Let $X \in \mathbb{Z}^2$ be non zero. Up to a replacement of X by $-X$, we can assume that $\zeta \ge 0$. The vector MX having non zero integer coordinates, we have:

$$r\zeta + |\varphi| + \lambda r\zeta + |\psi| \ge |r\zeta + \varphi| + |\lambda r\zeta + \psi| \ge 1.$$

Thus:

$$r\zeta+\varphi \ge \frac{1}{1+\lambda} - \frac{1}{1+\lambda}((2+\lambda)|\varphi|+|\psi|), \quad \lambda r\zeta+\psi \ge \frac{\lambda}{1+\lambda} - \frac{1}{1+\lambda}(\lambda|\varphi|+(1+2\lambda)|\psi|).$$

As $\max(|\varphi|, |\psi|) \le C'^{-1}\delta^{-1}\gamma^{-n}\|X\|$, there exists $c > 0$ such that if $n \ge c\ln\|p\|$ then $r\zeta + \varphi > 0$ and $\lambda r\zeta + \psi > 0$, so that $MX \in \mathbb{R}^2_{+*}$. □

Since the matrices are positive, there is $\gamma > 1$ such that $\|A_1^n\| \ge C\gamma^n$. For every cone $\mathcal{C}$ strictly contained in the positive cone, there exists a constant $c > 0$ such that, for every vector $q \in \mathbb{Z}^2_+$ belonging to $\mathcal{C}$,

$$\|A_1^n\| \, \|q\| \ge \|A_1^n q\| \ge c\|A_1^n\| \, \|q\|. \tag{42}$$

Corollary 2. *Conditions 1 and 2 are satisfied when $\mathcal{A} \subset \mathrm{SL}(2, \mathbb{Z}_+)$.*

Proof. Let n be the integer part of $c\log\|q\| + 1$ and $r > n$. Because of Lemma (3.2), $A_{\ell+r-n+1}^{\ell+r} q$ belongs to $\mathbb{Z}^2_+$ or $\mathbb{Z}^2_-$. Using (42), this gives for positive constants c_1, c_2:

$$\begin{aligned} \|A_1^{\ell+r} q\| &= \|A_1^{\ell+r-n} A_{\ell+r-n+1}^{\ell+r} q\| \ge c\|A_1^{\ell+r-n}\| \, \|A_{\ell+r-n+1}^{\ell+r} q\| \ge c\|A_1^{\ell+r-n}\| \\ &\ge c_1\|A_1^\ell\|\|A_{\ell+1}^{\ell+r-n}\| \ge c_2\gamma^{r-c\log\|p\|}\|A_1^\ell\| = c_2\gamma^r \, \|q\|^{-\delta} \, \|A_1^\ell\|. \end{aligned}$$

□

Using the version 6 of Condition 1, one deduces the following theorem:

Theorem 3.3. *Let (A_k) be a sequence of matrices in* $\mathrm{SL}(2, \mathbb{Z}_+)$*. The CLT holds for $f \in \mathcal{H}_0(\mathbb{T}^d)$ if $\liminf_n \frac{1}{n} \|\sum_1^n f(A_1^k .)\|_2^2 > 0$. In the stationary case either for μ-almost $\omega \in \Omega$, $(\|S_n(\omega, f)\|_2)$ is bounded or, for μ-almost $\omega \in \Omega$, the sequence $(n^{-\frac{1}{2}} \|S_n(\omega, f)\|_2)$ has a limit $\sigma(f) > 0$ not depending on ω. In the latter case, the CLT holds for a.e. ω.*

Remark 3. For instance (cf. Remark 2), if the sequence (A_n) is generated by an ergodic rotation on the circle, with $A(\omega) = A$ on an interval and $= B$ on the complementary, then we obtain the CLT for every such sequence.

3.3. **Example 2: Kicked sequences.** Let H be a hyperbolic matrix in $\mathrm{SL}(2, \mathbb{Z})$ and (B_n) be a sequence in $\mathrm{SL}(2, \mathbb{Z})$ such that the sequence $(\mathrm{trace}(B_n))$ is bounded. Let $s \geq 1$ be a fixed integer. Let us consider the "kicked" sequence of maps of the torus $\mathbb{T}^2$ defined by: $x \mapsto {}^t M_n x$ where

$$M_n = B_1 H^s ... B_{n-1} H^s B_n H^s. \tag{43}$$

L. Polterovich and Z. Rudnick defined the "stable mixing" for H as the property that, for every sequence (B_k) with bounded trace, there exists s_0 such that the sequence defined by (43) is mixing for every $s \geq s_0$, i.e., $\lim_n \int_{\mathbb{T}^d} f({}^t M_n x)\, \overline{g}(x)\, d\mu(x) = 0, \forall f, g \in L_0^2(\mathbb{T}^d)$. They proved that H is stable mixing if and only if H is not conjugate to its inverse. Their main tool is the notion of quasi-morphism.

Definition 3.4. A real function ρ on $\mathrm{SL}(2, \mathbb{Z})$ is a *homogeneous quasi-morphism* if

$$c(\rho) := \sup_{A, B \in \mathrm{SL}(2,\mathbb{Z})} |\rho(AB) - (\rho(A) + \rho(B))| < +\infty \tag{44}$$

and $\rho(A^n) = n\rho(A), \forall n \geq 0, \forall A \in \mathrm{SL}(2, \mathbb{Z})$.

Proposition 10. *([15]) 1) There exists $c > 0$ such that for every vector $p \in \mathbb{Z}^2 - \{0\}$ and every $A \in \mathrm{SL}(2, \mathbb{Z})$,*

$$\|Ap\| \geq c\, e^{|\rho(A)|/c(\rho)|}\, \|p\|^{-1}. \tag{45}$$

2) For every hyperbolic matrix $H \in \mathrm{SL}(2, \mathbb{Z})$ not conjugate to its inverse, there exists a homogeneous quasi-morphism ρ such that $\rho(H) = 1$ and ρ vanishes on all parabolic elements. It is bounded on any set of matrices in $\mathrm{SL}(2, \mathbb{Z})$ with bounded trace.

In the following we will consider the model of "kicked" systems for a sequential product of matrices and in the framework of stationary processes.

Let H be in $\mathrm{SL}(2, \mathbb{Z})$ *hyperbolic and not conjugate to its inverse.* We consider sequences (A_k) with values in a set $\mathcal{A}$ of matrices in $\mathrm{SL}(2, \mathbb{Z})$ of the form

$$\mathcal{A} = \{H, B_1, ..., B_j, ...\}, \text{with } \sup_j \mathrm{trace}(B_j) < +\infty. \tag{46}$$

The occurrence of the B_k's in the sequence (A_k) can be interpreted as a perturbation of the sequence $A_k = H, \forall k \geq 1$. We have stability if decorrelation and CLT still hold under small perturbations. "Smallness" means that the density of occurrence of the B_k's is small.

Condition 8. *The sequence (A_k) satisfies the perturbation condition $P(\varepsilon, r_0)$ for $\varepsilon > 0$ and $r_0 \geq 1$, if*

$$\sum_{k=\ell}^{\ell+r-1} 1_{A_k \neq H} \leq r\varepsilon, \ \forall r \geq r_0, \forall \ell \geq 1. \tag{47}$$

Proposition 11. *If a sequence (A_k) with values in $\mathcal{A}$ satisfies $P(\varepsilon, r_0)$ of Condition 8 for $\varepsilon > 0$ small enough, there is $\gamma > 1$ such that, for every vector p in $\mathbb{Z}^2 - \{0\}$:*

$$\|A_\ell^{\ell+r} p\| \geq c\gamma^r \, \|p\|^{-1}, \ \forall r \geq r_0(\varepsilon), \forall \ell \geq 1. \tag{48}$$

Proof. Let $r \geq r_0(\varepsilon)$. A product $A_\ell^{\ell+r} = A_\ell ... A_{\ell+r-1}$ reads $H^{k_1} B_1 H^{k_2} B_2 ... H^{k_t} B_t$ with $r = \sum_{i=1}^{\ell} k_i + t$ and $t = \sum_\ell^{\ell+r} 1_{A_k \neq H} \leq r\varepsilon$. Let ρ with $\rho(H) = 1$ be a quasi-morphism as given by Proposition 10. Let $C = \sup_i |\rho(B_i)|$. For a fixed $\lambda \in]0,1[$ we have:

$$\begin{aligned} |\rho(H^{k_1} B_1 H^{k_2} B_2 ... H^{k_t} B_t| &\geq \sum_{i=1}^{t} [k_i - \rho(B_i) - c(\rho)] \\ &= r - t - \sum_{i=1}^{t} [\rho(B_i) + c(\rho)] \\ &\geq r - t(1 + C + c(\rho)) \geq \lambda r, \end{aligned}$$

if $\varepsilon \leq (1-\lambda)(1 + C + c(\rho))^{-1}$. Hence, by (45) we obtain with $\gamma = e^{\lambda / c(\rho)} > 1$:

$$\|A_\ell^{\ell+r} p\| \geq c e^{\lambda r / c(\rho)} \, \|p\|^{-1} \geq c\gamma^r \, \|p\|^{-1}.$$

□

Now we consider stationary processes. Let (Ω, μ, θ) be a measure preserving ergodic dynamical system. Let $\omega \to A(\omega)$ be a measurable map from Ω to a set $\mathcal{A}$ of the form (46). The corresponding stationary process $(A_k(\omega)) = (A(\theta^k \omega))$ will be called stationary kicked process. We are going to study the behavior of the product $M_n(\omega) = A_1^n(\omega) = A(\omega) ... A(\theta^{n-1}\omega)$. First we give examples of stationary kicked processes satisfying (48).

Examples

1) If the set of matrices $\mathcal{A} = \{H^s B_j, s \in \mathbb{N}, j = 1, 2, ...\}$ is such that $\{B_j\}$ is a family of matrices in $\mathrm{SL}(2, \mathbb{Z})$ with bounded trace and H is hyperbolic, then for s big enough (48) holds. This example is valid for any dynamical system.

2) Another construction uses as dynamical system a subshift of finite type. Let be given the set of matrices $\{H^s, B_1, ..., B_r\}$. We can consider the set $\mathcal{A}$ as the set of states of a subshift of finite type. Suppose that the allowed transitions from a state B_r are necessarily to H^s. Then we obtain a kicked stationary system which satisfies Condition 8, if s is big enough.

3) Consider now a set $\mathcal{A}$ of the form (46). Suppose Ω is a compact metric space, (Ω, μ, τ) is a strictly ergodic dynamical system (for instance an ergodic rotation on a compact abelian group). Let $\omega \mapsto A(\omega) \in \mathcal{A}$ be a map from Ω to $\mathcal{A}$, with $A(\omega) \neq H$ on a set E. We suppose that $\mu(\partial E) = 0$. Then we have, uniformly in ω, $\lim_n \frac{1}{n} \sum_0^{n-1} 1_E(\tau^k \omega) = \mu(E)$. Therefore, if $\mu(E) > 0$, there is r_0 such that, for $r \geq r_0$ and every $\omega \in \Omega$, $\sum_0^{r-1} 1_E(\tau^k \omega) < 2r\mu(E)$.

For the kicked stationary processes defined by $A_k(\omega) = H$, if $\tau^k \omega \notin E$, $= B_k$, for one of the B_k's, if $\tau^k \omega \notin E$, then Condition 8 holds when $\mu(E)$ is small enough.

Remark that, if H is as above and ρ a quasi-morphism such that $\rho(H) = 1$, we can use several hyperbolic matrices H_i such that $\rho(H_i) > 0$ For instance we can take for H_i matrices which are conjugate to H (so that $\rho(H_i) = \rho(H) = 1$).

Multiplicative Ergodic Theorem

Under the hypothesis of Proposition 11, Condition 1 is satisfied. Another consequence is the positivity of the Lyapunov exponent. We will show how it can be used to obtain a weak form of the "frequencies separation" and finally a CLT. For it, we need the Multiplicative Ergodic Theorem of Oseledets ([13]) and some consequences of it.

Let $M_n(\omega)$ be a product $M_n(\omega) = A_1^n(\omega) = A(\omega)...A(\theta^{n-1}\omega)$, where $(A(\theta^k\omega))_{k\geq 0}$ is a stationary sequence of 2×2-matrices in $\mathrm{SL}(2,\mathbb{Z})$ with positive Lyapunov exponent α.

According to the Multiplicative Ergodic Theorem, there is a reduction of the form $M_n(\omega) = \Phi(\omega)^{-1}\Lambda(\omega)\Phi(\theta^n\omega)$, where $\Lambda(\omega)$ is a diagonal matrix. More precisely, the following proposition holds (cf. [16] for details):

Proposition 12. *There exists a matrix valued measurable function* Φ,

$$\Phi(\omega) = \begin{pmatrix} a(\omega) & b(\omega) \\ c(\omega) & d(\omega)\end{pmatrix},\ det\,\Phi(\omega) = 1,$$

such that, for every $n \geq 1$,

$$M_n(\omega)X = \lambda_n(\omega)\langle\varphi(\theta^n\omega), X\rangle\, U(\omega) + \lambda_n^{-1}(\omega)\langle\psi(\theta^n\omega), X\rangle\, V(\omega). \tag{49}$$

where $\varphi(\omega)$ *and* $\psi(\omega)$ *are linear forms and* $U(\omega), V(\omega)$ *vectors given by*

$$\langle\varphi(\omega), X\rangle = a(\omega)x + b(\omega)y,\ \langle\psi(\omega), X\rangle = c(\omega)x + d(\omega)y,$$
$$U(\omega) = \begin{pmatrix} d(\omega) \\ -c(\omega)\end{pmatrix}, V(\omega) = \begin{pmatrix} -b(\omega) \\ a(\omega)\end{pmatrix}.$$

$\lambda_n(\omega)$ *is a product* $\lambda_n(\omega) = \prod_{k=0}^{n-1}\lambda(\theta^k\omega)$ *and satisfies for a.e.* ω:

$$\frac{1}{n}\ln\lambda_n(\omega) = \frac{1}{n}\sum_0^{n-1}\ln\lambda(\theta^k\omega) \to \alpha,\ \frac{1}{n}\ln\lambda_n(\theta^{-n}\omega) = \frac{1}{n}\sum_1^{n}\ln\lambda(\theta^{-k}\omega) \to \alpha. \tag{50}$$

Moreover, writing $|\Phi(\omega)| = |a(\omega)| + |b(\omega)| + |c(\omega)| + |d(\omega)|$, *we have*

$$|\Phi(\omega)|^{-1} \leq \|U(\omega)\|, \|V(\omega)\| \leq |\Phi(\omega)| \tag{51}$$

and there is, for every $\varepsilon > 0$, *an a.e. finite function* $L(\varepsilon,\omega)$ *such that:*

$$L(\varepsilon,\omega)^{-1}\, e^{-\varepsilon|n|} \leq \|U(\theta^n\omega)\|, \|V(\theta^n\omega)\|, |\Phi(\theta^n\omega)| \leq L(\varepsilon,\omega)\, e^{\varepsilon|n|},\ \forall n \in \mathbb{Z}. \tag{52}$$

Let $\delta > 1$ and $\varepsilon > 0$ be two constants. By positivity of α and (50), there are a.e. finite positive functions $C(\omega), c(\omega)$ depending on δ such that

$$c(\omega)e^{\,n\delta^{-1}\alpha} \leq \lambda_n(\omega), \lambda_n(\theta^{-n}\omega) \leq C(\omega)e^{n\delta\alpha},\ \forall n \geq 1. \tag{53}$$

By (49) we have: $M_n(\omega)X = A_n(\omega, X) + B_n(\omega, X)$ with

$$A_n(\omega, X) = \lambda_n(\omega)\langle\varphi(\theta^n\omega), X\rangle\, U(\omega),\ B_n(\omega, X) = \lambda_n^{-1}(\omega)\langle\psi(\theta^n\omega), X\rangle\, V(\omega).$$

Inequalities (51), 52) and (53) imply for $\varepsilon > 0$ and $\alpha_1 = \delta^{-1}\alpha - \varepsilon$:

$$\begin{aligned}\|B_n(\omega, X)\| &\leq c(\omega)^{-1}\, e^{-\delta^{-1}\alpha n}|\Phi(\omega)|\, |\Phi(\theta^{-n}\omega)|\, \|X\| \\ &\leq c(\omega)^{-1}L(\varepsilon,\omega)\, |\Phi(\omega)|\, e^{-\alpha_1 n}\, \|X\|;\end{aligned}$$

hence $\|B_n(\omega, X)\| \leq \frac{1}{2}$, for $n \geq s(\omega, X)$, where $s(\omega, X) \in \mathbb{N}$ is such that

$$s(\omega, X) \in [S(\omega) + \frac{1}{\alpha_1}\ln\|X\| - 1,\ S(\omega) + \frac{1}{\alpha_1}\ln\|X\|]$$

and $S(\omega) = \frac{1}{\alpha_1}\ln(2c(\omega)^{-1}\,L(\frac{\alpha}{2},\omega)\,|\Phi(\omega)|) + 1$.

Remark that if A and B are two vectors with $\|A+B\| \geq 1$ and $\|B\| \leq \frac{1}{2}$, then

$$\frac{1}{2}\|A+B\| \leq \|A\| \leq \frac{3}{2}\|A+B\|.$$

Therefore, if X is a non zero vector in $\mathbb{Z}^2$, we have, as $\|M_n(\omega)X\| \geq 1$,

$$\frac{1}{2}\,\|M_n(\omega)X\| \;\leq \|A_n(\omega,X)\| \leq\; \frac{3}{2}\,\|M_n(\omega)X\|,\;\forall n \geq s(\omega,X).$$

Likewise, (53) implies

$$\begin{aligned}\|B_n(\theta^{-n}\omega,X)\| &= \lambda_n^{-1}(\theta^{-n}\omega)|\langle\psi(\omega),X\rangle|\|V(\theta^{-n}\omega)\|\\ &\leq c(\omega)^{-1}\,e^{-\delta^{-1}\alpha n}|\Phi(\omega)|\,|\Phi(\theta^{-n}\omega)|\,\|X\|.\end{aligned}$$

Since this is the same bound as for $\|B_n(\omega,X)\|$, it follows $\|B_{s(\omega,X)}(\theta^{-s(\omega,X)}\omega,X)\| \leq \frac{1}{2}$.

Therefore, as $\|A_n(\theta^{-n}\omega,X) + B_n(\theta^{-n}\omega,X)\| = \|M_n(\theta^{-n}\omega)X\| \geq 1$, we get:

$$\|A_{s(\omega,X)}(\theta^{-s(\omega,X)}X)\| > \frac{1}{2}.$$

This implies

$$|\langle\varphi(\omega),X\rangle| \geq \frac{1}{2}\lambda^{-1}_{s(\omega,X)}(\theta^{-s(\omega,X)}\omega)\,\|U(\theta^{-s(\omega,X)}\omega)\|^{-1};$$

hence using (53), with the a.e. positive function

$$T(\omega) := \frac{1}{2}C(\omega)^{-1}\,L(\varepsilon,\omega)^{-1}\,e^{-(\delta\alpha+\varepsilon)S(\omega)}$$

$$|\langle\varphi(\omega),X\rangle| \geq T(\omega)\|X\|^{-(1+\varepsilon)}. \tag{54}$$

Now let X be a non zero vector in $\mathbb{Z}^2$. We can compare $\|M_{n-r}(\omega)Y\|$ and $\|M_n(\omega)X\|$. Let n,r be such that $n \geq s(\omega,X)$ and $0 < r \leq n$. We have, for every vector Y,

$$\begin{aligned}\frac{\|M_{n-r}(\omega)Y\|}{\|M_n(\omega)X\|} &\leq \frac{3}{2}\,\frac{\lambda_{n-r}(\omega)}{\lambda_n(\omega)}\,[\frac{|\langle\varphi(\theta^{n-r}\omega),Y\rangle|\,\|U(\omega)\|}{|\langle\varphi(\theta^{n}\omega),X\rangle|\,\|U(\omega)\|} + \frac{|\langle\psi(\theta^{n-r}\omega),Y\rangle|\,\|V(\omega)\|}{|\langle\varphi(\theta^{n}\omega),X\rangle|\,\|U(\omega)\|}]\\ &\leq \frac{3}{2}\,\lambda_r^{-1}(\theta^{n-r}\omega)\,(1+\frac{\|V(\omega)\|}{\|U(\omega)\|})\,\frac{|\Phi(\theta^{n-r}\omega)|}{|\langle\varphi(\theta^n\omega),X\rangle|}\,\|Y\|\\ &= \frac{3}{2}\rho_2(r,\theta^n\omega,X)\,(1+\frac{\|V(\omega)\|}{\|U(\omega)\|})\,\|Y\|,\end{aligned}$$

where

$$\begin{aligned}\rho_2(r,\omega,X) &:= \lambda_r^{-1}(\theta^{-r}\omega)\,\frac{|\Phi(\theta^{-r}\omega)|}{|\langle\varphi(\omega),X\rangle|}\\ &\leq c(\omega)^{-1}\,e^{-\delta^{-1}\alpha r}\,L(\varepsilon,\omega)\,e^{\varepsilon r}\,T(\omega)^{-1}\,\|X\|^{1+\varepsilon}.\end{aligned}$$

Take $R_2(\omega) = \frac{3}{2}(1+\frac{\|V(\omega)\|}{\|U(\omega)\|})$ and $R_1(\omega) = c(\omega)^{-1}L(\varepsilon,\omega)\,T(\omega)^{-1}$. Using (54), we obtain:

$$\frac{\|M_{n-r}(\omega)Y\|}{\|M_n(\omega)X\|} \leq R_1(\theta^n\omega)\,R_2(\omega)\,e^{-(\delta^{-1}-\varepsilon)\alpha r}\,\|X\|^{1+\varepsilon}\,\|Y\|,$$

for all $n \geq S(\omega) + \frac{1}{\alpha_1}\ln\|X\|$. By setting $\delta_1(\omega) = R_1(\omega)^{-1}$ and $\delta_2(\omega) = R_2(\omega)^{-1}$, we obtain with α the Lyapunov exponent of (M_n):

Proposition 13. *Let $0 < \alpha_1 < \alpha$ and $\varepsilon > 0$. There are a.e. positive finite functions δ_1, δ_2 and S such that, for every $X \in \mathbb{Z}^2 - \{0\}$, for every $n \geq S(\omega) + \frac{1}{\alpha_1} \ln \|X\|$, for every $0 \leq r \leq n$,*

$$\|M_n(\omega)X\| \geq \delta_1(\theta^n\omega)\, \delta_2(\omega)\, e^{\alpha_1 r}\, \|X\|^{-(1+\varepsilon)}\, \|M_{n-r}(\omega)\|. \tag{55}$$

Remark. The proposition is valid for $d > 2$ if the Lyapunov exponents are < 1, except the largest > 1.

Theorem 3.5. *The CLT is satisfied by a kicked stationary process $(A_k(\omega))$ under Condition 8.*

Proof. As Condition 1 (and its variant Condition 6) follows from (48), by Proposition 8 the variance $\sigma^2(f)$ exists and, for a.e. ω, $\lim \frac{1}{n}\| \sum_1^n f({}^t\!A_1^k(\omega).)\|_2^2 = \sigma^2(f)$. Assume that f is not a coboundary and therefore $\sigma(f) > 0$.

Let $\varepsilon > 0$. Consider the positive functions δ_1, δ_2 on Ω given by Proposition 13 applied to $M_k(\omega) = A_1^k(\omega)$. Let $c > 0$ be a constant and $F_i := \{\omega : \delta_i(\omega) > c^{\frac{1}{2}}\}$, for $i = 1, 2$. Let $J = J(\omega, \varepsilon)$ be the sequence of positive integers defined by $n \in J \Leftrightarrow \theta^n(\omega) \in F_1$. If c is small enough, then $\mu(F_1), \mu(F_2) > 1 - \varepsilon$. By the ergodic theorem, for a.e. ω, the asymptotic density of $J(\omega, \varepsilon)$ is bigger than $1 - \varepsilon$.

For $\omega \in F_2$, by (55) of Proposition 13, Condition 7 is satisfied along the subsequence J, since for $0 < \alpha_2 < \alpha_1$ (where α_1 is the constant in (55)), there is $C_1 > 0$ such that

$$\|A_1^n(\omega)q\| \geq c\, e^{\alpha_2 r}\, \|A_1^{n-r}(\omega)\|, \forall\, n \geq r \geq C_1 \ln \|q\|,\, q \in \mathbb{Z}^2 - \{0\},\, \forall n \in J(\omega, \varepsilon).$$

Let $S_n^J(\omega, f)(.) := \sum_{1 \leq k \leq n,\, k \in J} f({}^t\!A_1^k(\omega).)$. For ε small enough, Proposition 3 implies $\|S_n^J(\omega, f)\|_2^2 \geq C\, \mathrm{Card}([1, n] \cap J)$ for $C > 0$, for n big enough.

Therefore we can apply Theorem 2.3. We obtain the convergence in distribution toward the normal law of the sequence $(\frac{S_n^J(\omega,f)}{\|S_n^J(\omega,f)\|_2})$. Since by Proposition 3, $\frac{1}{n}\|S_n(\omega, f) - S_n^J(\omega, f)\|_2^2 < \varepsilon$ for n big, this implies $\lim_n |\mathbb{E}[e^{it\frac{S_n(\omega,f)}{\|S_n(\omega,f)\|_2}}] - e^{-\frac{1}{2}t^2}| = 0$ for every t. $\square$

3.4. **Endomorphisms and the coboundary condition.** Let $(A_n)_{n\geq 0}$ be a sequence in $\mathcal{M}_d^*(\mathbb{Z})$, with $A_0 = \mathrm{Id}$, and let $\tau_n : x \mapsto^t A_n x \mod 1$ be the corresponding of endomorphisms of $\mathbb{T}^d$. We consider the decreasing family of σ-algebras $(\mathcal{B}_n)_{n\geq 1} = (\tau_1^{-1}\tau_2^{-1}...\tau_n^{-1}\mathcal{B})_{n\geq 1}$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{T}^1$.

Let Γ_n be the subgroup of $\mathbb{T}^d$ defined as the kernel $\Gamma_n = \{z \in \mathbb{T}^d : A_n...A_1 z = 0 \mod 1\}$. Then $\mathcal{B}_n$ is the σ-algebra of the Γ_n-invariant Borel sets and the exactness property $\cap_n \mathcal{B}_{n\geq 1} = \mathcal{B}_0$ (the trivial σ-algebra of $\mathcal{B}$) is equivalent to the density in $\mathbb{T}^d$ of the group $\cup_n \Gamma_n$.

Suppose that $A_n = q_n B_n$, with q_n an integer > 1 and $B_n \in \mathrm{SL}(d, \mathbb{Z})$. The exactness property holds and we can use a martingale method to show a CLT. To simplify, we present the case $d = 1$, $\tau_n x = q_n x \mod 1$.

This is a special case of a more general setting using β-transformations presented in [9]. We recall briefly the method. Let f be a Hölder function on $\mathbb{T}^1$ with $\lambda(f) = 0$ and let $S_n f(x) = \sum_{k=0}^{n-1} f(q_k...q_1 q_0 x)$ be the ergodic sums.

Let $T_n f = f \circ \tau_n$, $Q_n f(x) = q_n^{-1} \sum_{j=0}^{q_n - 1} f(x + \frac{j}{q_n})$. We defined h_n by the relations $h_{n+1} = Q_{n+1} f + Q_{n+1} h_n$, with $h_0 = 0$.

$$h_n \quad = \quad Q_n f + Q_n Q_{n-1} f + ... + Q_n Q_{n-1}...Q_1 f.$$

$Q_n Q_{n-1} ... Q_1 f$ is uniformly exponentially close to the integral of f, hence exponentially small, so that (h_n) is uniformly bounded. We write

$$\varphi_n = f + h_n - T_{n+1} h_{n+1},\ U_n = T_1 ... T_n \varphi_n.$$

(U_n) is a sequence of differences of reversed martingale for the filtration $(\mathcal{B}_n)_{n\geq 1}$. According to the relation

$$\sum_0^{n-1} T_1 ... T_k f = \sum_0^{n-1} U_k + T_1 \cdots T_n h_n$$

we can replace $\sum_0^{n-1} T_1 ... T_k f_k$ by the reversed martingale $\sum_0^{n-1} U_k$ with a bounded error term and apply the CLT theorem of B.M. Brown for martingales ([6]). We obtain that either the norms $\|S_n f\|_2$ are bounded (in that case it can be shown that the sequence $\sum_{k=0}^{n-1} f(\tau_k \cdots \tau_1 x), n \geq 1$ is bounded for a.e. x), or the sequence $(\frac{f + T_1 f + ... + T_1 .. T_{n-1} f}{\|S_n f\|_2})_{n\geq 1}$ converges in distribution to $N(0,1)$.

Now we consider the stationary model to study more precisely the question of degeneracy in the CLT. With the notations 3.1 let $(\Omega \times \mathbb{T}^1, \theta_\tau, \mu \times dx)$ be a skew product where θ is invertible and τ takes only values 2 and 3 with positive measure (clearly the results remains true if we replace 2 and 3 by two other relatively prime numbers). In this case Condition 1 is obviously satisfied. Let f be in $\mathcal{H}_0(\mathbb{T}^1)$. Then either for a.e. ω the CLT holds with a variance $\sigma^2(f) > 0$ or f is a coboundary for θ_τ.

In the second case there exists a function g in $L^2(\Omega \times \mathbb{T}^1)$ such that, for almost every (x, ω), $f(x) = g(\omega, x) - g(\theta\omega, \tau(\omega)x)$. For almost ω, $x \mapsto g(\omega, x)$ is in $L^2(\mathbb{T}^1)$.

Theorem 3.6. *Let f be a non zero function on $\mathbb{T}^1$ in $\mathcal{H}_0$. If for every integer $L > 0$ there exists $a_0, \ldots, a_L$ such that the sets*

$$[a_0 \ldots a_L r] := \{\omega :\ \tau(\theta^i \omega) = a_i, i = 0, \ldots, L, \tau(\theta^{L+1}\omega) = r\}$$

for $r = 2, 3$ are both of positive measure, then f is not a coboundary for θ_τ.

Proof. Let f be in $\mathcal{H}_0$ and g in $L^2(\Omega \times \mathbb{T}^1)$ such that, for almost every (x, ω), $f(x) = g(\omega, x) - g(\theta\omega, \tau(\omega)x)$. Let $f(x) = \sum_{k\in\mathbb{Z}} f_k \chi_k,\ g(\omega, x) = \sum_{k\in\mathbb{Z}} g_k(\omega)\chi_k$, with $\sum_{k\in\mathbb{Z}} |f_k|^2 < \infty$, $\sum_{k\in\mathbb{Z}} |g_k(\omega)|^2 < \infty$, be the Fourier series of f, g. The equality

$$\sum_{k\in\mathbb{Z}} f_k \chi_k = \sum_{k\in\mathbb{Z}} g_k(\omega)\chi_k - \sum_{k\in\mathbb{Z}} g_k(\theta\omega)\chi_k(\tau(\omega)\cdot).$$

gives relations between the sequences (f_k) and $(g_k(\omega))$:
– $f_\ell = g_\ell(\omega)$, if $\tau(\omega)$ does not divide ℓ
– $f_\ell = g_\ell(\omega) - g_{\frac{\ell}{\tau(\omega)}}(\theta\omega)$, if $\tau(\omega)$ divides ℓ.
With the convention $f_q = 0$, $g_q = 0$ if q is not an integer, we always have

$$f_\ell = g_\ell(\omega) - g_{\frac{\ell}{\tau(\omega)}}(\theta(\omega)). \tag{56}$$

This relation can be written

$$f_{\tau(\theta^{-1}\omega)k} = g_{\tau(\theta^{-1}\omega)k}(\theta^{-1}\omega) - g_k(\omega). \tag{57}$$

Iterating (57) and summing the equalities, we obtain:

$$\sum_{\ell=1}^{L} f_{\tau(\theta^{-\ell}\omega)\ldots\tau(\theta^{-1}\omega)k} = g_{\tau(\theta^{-L}\omega)\ldots\tau(\theta^{-1}\omega)k}(\theta^{-L}\omega) - g_k(\omega).$$

Remark that, as f is in $\mathcal{H}_0$, there exists $\alpha > 0$ such that $|f_k| \leq C|k|^{-\alpha}$. As τ takes values in $\{2, 3\}$, this implies the the series $\sum_{\ell=1}^{\infty} f_{\tau(\theta^{-\ell}\omega)\ldots\tau(\theta^{-1}\omega)k}$ converges when $k \neq 0$ and the sequence $(g_{\tau(\theta^{-L}\omega)\ldots\tau(\theta^{-1}\omega)k}(\theta^{-L}\omega))_{L\geq 1}$ converges. But, for almost every ω, $g_\ell(\omega)$ tends to 0 when $|\ell|$ tends to infinity. Thus, almost surely, the sequence $(g_{\tau(\theta^{-L}\omega)\ldots\tau(\theta^{-1}\omega)k}(\theta^{-L}\omega))_{L\geq 1}$ has a subsequence tending to 0 and consequently tends to 0. So almost surely in ω we have

$$g_k(\omega) = -\sum_{\ell=1}^{\infty} f_{\tau(\theta^{-\ell}\omega)\ldots\tau(\theta^{-1}\omega)k}. \tag{58}$$

Iterating (56) in a similar way we obtain:

$$g_k(\omega) = f_k + \sum_{\ell=0}^{\infty} f_{\frac{k}{\tau(\omega)\ldots\tau(\theta^{\ell}\omega)}}. \tag{59}$$

In particular a necessary condition for f to be a coboundary is

$$\sum_{\ell=1}^{\infty} f_{\tau(\theta^{-\ell}\omega)\ldots\tau(\theta^{-1}\omega)k} + f_k + \sum_{\ell=0}^{\infty} f_{\frac{k}{\tau(\omega)\ldots\tau(\theta^{\ell}\omega)}} = 0, \text{ for a.e. } \omega.$$

Suppose that f is a trigonometric polynomial with $\deg(f) \leq D$. Let L be an integer such that $D < 2^L$, $a_0, \ldots, a_L$ be such that $[a_0 \ldots a_L 2]$ and $[a_0 \ldots a_L 3]$ have positive measures, and k be an integer coprime with 2 and 3. From (58) and $D < 2^L$, for almost every ω we have

$$g_{a_0\ldots a_L 2k}(\omega) = g_{a_0\ldots a_L 3k}(\omega) = 0.$$

On the other hand, using (59), we have:

$$\begin{aligned} 0 &= g_{a_0\ldots a_L 2k}(\omega) = f_{a_0\ldots a_L 2k} + f_{a_1\ldots a_L 2k} + \ldots + f_{2k} + f_k, \text{ if } \omega \in [a_0 \ldots a_L 2], \\ &= g_{a_0\ldots a_L 2k}(\omega) = f_{a_0\ldots a_L 2k} + f_{a_1\ldots a_L 2k} + \ldots + f_{2k} + f_{2k/3}, \text{ if } \omega \in [a_0 \ldots a_L 3]. \end{aligned}$$

But $f_{2k/3}$ is zero so that $f_k = 0$. Suppose now that $f_{3^j k} = 0$ for $j \leq J$. We have:

$$\begin{aligned} 0 &= g_{a_0\ldots a_L 3^{J+2}k}(\omega) \\ &= f_{a_0\ldots a_L 3^{J+2}k} + f_{a_1\ldots a_L 3^{J+2}k} + \ldots + f_{3^{J+2}k} + f_{3^{J+2}k/2}, \text{ if } \omega \in [a_0 \ldots a_L 2], \\ &= f_{a_0\ldots a_L 3^{J+2}k} + f_{a_1\ldots a_L 3^{J+2}k} + \ldots + f_{3^{J+2}k} + f_{3^{J+1}k} + 0, \text{ if } \omega \in [a_0 \ldots a_L 3]. \end{aligned}$$

From $f_{3^{J+2}k/2} = 0$, we then deduce that $f_{3^{J+1}k} = 0$. Thus we have $f_{3^j k} = 0$ for every $j \geq 0$. We can show as well that $f_{2^j k} = 0$ for $j \geq 0$. Suppose now that $f_{2^j 3^\ell k} = 0$ for $j + \ell \leq n$ (this is true for $n = 0$). Let j, ℓ be such that $j + \ell = n + 1$. We have

$$\begin{aligned} &0 = g_{a_0\ldots a_L 2^{j+1}3^\ell k}(\omega) = f_{a_0\ldots a_L 2^{j+1}3^\ell k} + f_{a_1\ldots a_L 2^{j+1}3^\ell k} + \ldots + f_{2^j 3^\ell k} + 0, \\ &\quad \text{if } \omega \in [a_0 \ldots a_L 2], \\ &0 = g_{a_0\ldots a_L 2^{j+1}3^\ell k}(\omega) = f_{a_0\ldots a_L 2^{j+1}3^\ell k} + f_{a_1\ldots a_L 2^{j+1}3^\ell k} + \ldots + f_{2^{j+1}3^{\ell-1}k} + 0, \\ &\quad \text{if } \omega \in [a_0 \ldots a_L 3], \end{aligned}$$

so that $f_{2^{j+1}3^{\ell-1}k} = f_{2^j 3^\ell k}$. As $f_{2^{n+1}k} = 0$, this implies that for all $j = 0, \ldots, n+1$, we have $f_{2^j 3^{n+1-j}k} = 0$. We conclude that all the coefficients $f_{2^j 3^\ell k}$ vanish so that $f \equiv 0$.

Consider now the case when f is in $\mathcal{H}_0$. Let k be a non zero integer coprime with 2 and 3, $\varepsilon > 0$ a real number. Let L be such that for every word $b_0 \ldots b_L$ composed with letters 2 and 3 we have

$$|\sum_{\ell=1}^{\infty} f_{\tau(\theta^{-\ell}\omega)\ldots\tau(\theta^{-1}\omega)b_0\ldots b_L k}| < \varepsilon. \tag{60}$$

Let $a_0 \ldots a_L$ such that $[a_0 \ldots a_L 2]$ and $[a_0 \ldots a_L 3]$ have positive measures. Because of (60) and (58) we have $|g_{a_0\ldots a_L 2k}(\omega)|, |g_{a_0\ldots a_L 3k}(\omega)| < \varepsilon$. Using (59), we have:

$$\begin{aligned} g_{a_0\ldots a_L 2k}(\omega) &= f_{a_0\ldots a_L 2k} + f_{a_1\ldots a_L 2k} + \ldots + f_{2k} + f_k, \text{ if } \omega \in [a_0 \ldots a_L 2], \\ g_{a_0\ldots a_L 2k}(\omega) &= f_{a_0\ldots a_L 2k} + f_{a_1\ldots a_L 2k} + \ldots + f_{2k} + f_{2k/3}, \text{ if } \omega \in [a_0 \ldots a_L 3]. \end{aligned}$$

But $f_{2k/3}$ is zero, so that we have $|f_k| < 2\varepsilon$ for every $\varepsilon > 0$; hence $f_k = 0$. Reasoning as above, we prove by induction that for all j, k, $f_{2^j 3^l k} = 0$ and finally that f is zero. □

Remark that the condition of the theorem is satisfied if τ is totally ergodic, i.e., τ^k is ergodic for all $k \geq 1$, for example if τ is an irrational rotation or a mixing transformation.

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